Analogues of Picard sets for meromorphic functions with a deficient value

by Guy Kendall

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Abstract

Picard's theorem states that a non-constant function which is meromorphic in the complex plane $\mathbf{C}$ omits at most two values of the extended complex plane $\mathbf{C}^*$. A *Picard set* for a family of functions $\mathcal{F}$ is a subset E of the plane such that every transcendental $f \in \mathcal{F}$ takes every value of $\mathbf{C}^*$, with at most two exceptions, infinitely often in $\mathbf{C} - E$.

If f is transcendental and meromorphic in the plane, then:

(i) [Hayman and others] if N is a positive integer, $f^N f'$ takes all finite non-zero values infinitely often;

(ii) [Hayman] either f takes every finite value infinitely often, or each derivative $f^{(k)}$ takes every finite non-zero value infinitely often.

We can seek *analogues* of Picard sets ie subsets E of the plane and an associated family of functions $\mathcal{F}$, such that (for case (i)) $f^N f'$ takes all finite non-zero values infinitely often in $\mathbf{C} - E$, for all $f \in \mathcal{F}$. Similarly for case (ii).

In this thesis we improve or extend the results previously known, both for Picard sets proper and for the analogous cases (i) and (ii) mentioned above, when the family of functions $\mathcal{F}$ consists of meromorphic functions which have deficient poles (in the sense of Nevanlinna).

Contents

Chapter 1

Introduction

1.1 Background

The classical theorem of Picard states that a non-constant function f which is meromorphic (ie analytic except for isolated poles) in the complex plane $\mathbf{C}$ may omit at most two values of the extended complex plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$.

A standard example is the function $f(z) = e^z$, which omits the values 0, ∞, but takes every other value of $\mathbf{C}^*$ infinitely often.

Lehto [19] introduced the concept of a *Picard set.* A subset E of $\mathbf{C}$ is a Picard set for a family $\mathcal{F}$ of functions meromorphic in $\mathbf{C}$ if every transcendental $f \in \mathcal{F}$ takes every value of $\mathbf{C}^*$, with at most two exceptions, infinitely often in $\mathbf{C} - E$. [A function f is *transcendental* if is not *rational* ie there do not exist polynomials P, Q such that $f(z) = P(z)/Q(z)$.]

Thus, for example, the set $E = \{2in\pi : n \in \mathbf{Z}\}$ is *not* a Picard set for the family of entire functions, because the function $f(z) = e^z$ fails to take any of the three values 0, ∞, 1 on $\mathbf{C} - E$.

The existence of Picard sets for entire functions and for meromorphic functions (we shall throughout use "meromorphic" to mean "meromorphic in $\mathbf{C}$" unless stated otherwise) has been investigated extensively. The position for these two families has effectively been settled, as follows.

In [24], Toppila proved that a countable set of points $E = \{a_m\}_{m=1}^{\infty}$, where the a_m converge to infinity, is a Picard set for entire functions if there exists $\varepsilon > 0$ such that the a_m satisfy

$$|a_m - a_{m'}| > \frac{\varepsilon |a_m|}{\log |a_m|}, \quad m \neq m'. \tag{1.1}$$

In the same paper he exhibited an example to show that this condition is effectively best possible.

Baker and Liverpool [4] proved further that if the set $E = \{a_m\}$ above satisfies (1.1), then there exists a sequence of small radii $d_m \to 0$ such that the countable union of open discs

$$\cup_{m=1}^{\infty} B(a_m, d_m) = \cup_{m=1}^{\infty} \{z : |z - a_m| < d_m\} \tag{1.2}$$

is a Picard set for entire functions.

For meromorphic functions, Toppila proved in [22] that if $E = \{a_m\}$ is a countable set of points converging to infinity, which satisfy

$$|a_m|^2 = O(|a_{m+1}|),$$

then E is a Picard set for the family of meromorphic functions.

He further exhibited an example to show that this condition is essentially best possible.

In a clear departure from the case for entire functions, Toppila also proved [23] that no countable union of open discs tending to infinity can be a Picard set for the family of meromorphic functions.

Although these results settle the two basic cases discussed above, further investigation into Picard-type sets is possible in the following directions:

(i) we can aim to establish results on Picard sets for other families of functions; or

(ii) we can consider general theorems which have been established for the value distribution of meromorphic functions and their derivatives, and aim to establish analogues of Picard sets in these situations. We shall explain what we mean by this in Section 1.3.

In Chapter 2 we shall improve a result which has been obtained in direction (i). Chapters 3, 4 and 5 will be concerned with certain problems in direction (ii). Section 1.3 sets out in more detail the results we shall be aiming to prove, but first we need some more background theory.

1.2 Nevanlinna Theory

Underlying all of our investigations will be Nevanlinna theory, which was developed by R. Nevanlinna in the 1920s. A standard reference is Chapters 1-4 of Hayman's book [8]. We shall set out in this section the basics of this theory. Everything in this section is drawn from [8] and due to Nevanlinna unless otherwise indicated.

Let f be a non-constant meromorphic function. For $r \geq 0$ we define $n(r, f)$ to be the number of poles of f in $|z| \leq r$, where a pole of multiplicity p is counted p times. We denote by $\overline{n}(r, f)$ the corresponding quantity where each pole is counted once only, regardless of multiplicity.

We also define, for a complex number a,

$$n(r, a, f) = n(r, 1/(f - a)), \tag{1.3}$$

the number of times f takes the value a in $|z| \leq r$. (We also sometimes write $n(r, \infty, f)$ for $n(r, f)$.)

Next, we define

$$N(r, f) = \int_0^r \frac{n(t, f)}{t} dt, \tag{1.4}$$

the *integrated counting function* of f (slight modifications are required if $f(0) = \infty$; these are dealt with in [8] but we shall not discuss them here, as they do not cause significant problems to the theory).

We define $N(r, a, f)$ analogously, and $\overline{N}(r, f)$, $\overline{N}(r, a, f)$ similarly.

We define also

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \tag{1.5}$$

the *proximity function*, which is the average of $\log^+ |f(z)|$ on $|z| = r$, where

$$\log^+ x = \max(\log x, 0). \tag{1.6}$$

We define similarly

$$m(r,a,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta}) - a}\right| d\theta. \tag{1.7}$$

We set

$$T(r,f) = N(r,f) + m(r,f). \tag{1.8}$$

$T(r,f)$ is called the *characteristic function* of f. It is strictly increasing in r.

For a function f analytic on $|z| = r$, we define

$$M(r,f) = \max_{|z|=r} |f(z)|. \tag{1.9}$$

If f is entire, we have

$$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f), \quad 0 \le r < R. \tag{1.10}$$

Note therefore that for meromorphic f, we have $T(r,f) \to \infty$ as $r \to \infty$, since either f has a pole w, in which case $N(r,f) \ge \log r/|w|$, or f is entire, in which case $T(r,f) \ge \frac{1}{3}\log^+ M(\frac{1}{2}r, f) \to \infty$ by Liouville's theorem and the maximum modulus principle.

Nevanlinna's *first fundamental theorem* states that

$$N(r,a,f) + m(r,a,f) = T(r,f) + O(1) \tag{1.11}$$

for every complex number a.

We next define the (Nevanlinna) *deficiency* of a value a (finite or infinite) by (using the first fundamental theorem)

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}. \tag{1.12}$$

We also sometimes need to consider the *Valiron deficiency*

$$\Delta(a,f) = \limsup_{r\to\infty} \frac{m(r,a,f)}{T(r,f)} = 1 - \liminf_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}. \tag{1.13}$$

Nevanlinna's *second fundamental theorem* states that the number of values a in $\mathbf{C}^*$ for which $\delta(a,f) > 0$ is countable and that

$$\sum_{a\in\mathbf{C}^*} \delta(a,f) \leq 2. \tag{1.14}$$

[Actually the second fundamental theorem is stronger than this - see [8] section 2.4 - but the above is all we shall need here and so for simplicity we shall refer to this result as the second fundamental theorem.]

Note that this result contains Picard's theorem: if a non-constant meromorphic function f omits a value a then we have $n(r,a,f) = 0$ for all r, and hence $N(r,a,f) = 0$ for all r and so $\delta(a,f) = 1$. The second fundamental theorem tells us that there cannot be more than two such values.

There are a number of other useful definitions and results, as follows.

Suppose that a meromorphic function f is transcendental. Then we have

$$\frac{T(r,f)}{\log r} \to \infty \tag{1.15}$$

as $r \to \infty$.

We define the *order* of a function f, $\rho(f)$, by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \qquad (1.16)$$

We denote by $S(r,f)$ any term which is $O(\log(rT(r,f)))$ as $r \to \infty$, either through all values of r, if the order of f is finite, or outside a set of r of finite linear measure, if the order of f is infinite.

If f is transcendental then, from (1.15), $S(r,f) = o(T(r,f))$ as $r \to \infty$, outside the exceptional set (if any).

A useful result is the *lemma of the logarithmic derivative*, ie the fact that

$$m(r, f'/f) = S(r,f). \qquad (1.17)$$

1.3 Problems investigated in this thesis

We shall be concerned primarily with establishing analogues of Picard sets for meromorphic functions f with Nevanlinna deficient poles, ie $\delta(\infty, f) > 0$.

In [26], Toppila proved that a countable union of open discs $B(a_m, d_m)$ satisfying certain conditions on the spacing of the a_m and the size of the d_m forms a Picard set for the family of meromorphic functions f with deficient poles.

Since $\delta(\infty, f) > 0$ for such an f, Nevanlinna's second fundamental theorem (1.14) tells us that at most one finite value may be omitted completely by f. Toppila's result shows that the pre-image of at most two values may be contained in a Picard set as given in his theorem. This suggests the question, if two such values exist, of whether they may both be finite or whether one must necessarily be the deficient value ∞.

In this direction, Langley [18] proved that only one finite exceptional value is possible in this situation, although the spacing condition in his result was more restrictive than Toppila's (ie the a_m were required to be further apart in Langley's result).

In Chapter 2, we aim to close the gap between these results, by improving the spacing condition in Langley's theorem. We obtain (Theorem 5) a spacing condition and radii of the same form as Toppila's, although the position is left open for certain values of constants. We also show (Theorem 6) that the conditions in our result may be relaxed in the case when the points a_m all lie on a ray.

In Chapters 3 and 4 we consider analogues of Picard sets arising from the following theorem, which is due to Hayman for $N \geq 3$ [9], Mues for $N = 2$ [20], and Bergweiler and Eremenko for $N = 1$ [6]:

Theorem 1 *Let f be a transcendental meromorphic function, and let $N \geq 1$ be an integer. Then $f^N f'$ takes all finite non-zero complex values infinitely often.*

In the same way as for Picard sets proper, we can ask what subsets E of the plane exist such that, for some family $\mathcal{F}$ of transcendental meromorphic functions we can say that for any $f \in \mathcal{F}$, the function $f^N f'$ takes all finite non-zero values infinitely often on $\mathbf{C} - E$.

Anderson, Baker and Clunie [2] proved a result in this direction for entire functions, and also a result for meromorphic functions with a strong spacing condition and $N \geq 11$.

We prove in Chapter 3 results for $N \geq 2$ where E is a countable set of points (Theorem 28), and where E is a countable union of open discs (Theorem 29), for certain subsets of the family of transcendental meromorphic functions with deficient poles.

In the course of this, we prove a number of lemmas (Lemmas 42, 43 and 44) concerning circumstances in which we can conclude that $\delta(\infty, f^N f') > 0$ from the deficiency of f or f'.

To show that these lemmas are not redundant, we exhibit an example (Section 4.3) to show that it is possible to have $\delta(\infty, f) > 0$ and $\delta(\infty, f^N f') = 0$.

In Chapter 4, we also consider the case $N = 1$ and prove a result for a point set E, for transcendental entire functions (Theorem 54).

In Chapter 5 we consider analogues of Picard sets arising from the following theorem, which is due to Hayman [8, p60]:

Theorem 2 *Let f be a transcendental meromorphic function. Then either f takes every finite value infinitely often, or each of its derivatives $f^{(k)}$, $k \geq 1$ takes every finite non-zero value infinitely often.*

We ask what subsets E of the plane exist such that, for some family $\mathcal{F}$ of transcendental meromorphic functions we can say that for any $f \in \mathcal{F}$, either f takes all finite values infinitely often on $\mathbf{C} - E$ or each of its derivatives $f^{(k)}$

takes every finite non-zero value infinitely often on $\mathbf{C} - E$.

Langley proved in [13, p17,57] and [16] results in this direction for:

(i) E a countable union of open discs, for entire functions; and

(ii) E a countable set of points, for meromorphic functions, with $k = 1$ fixed, a strong spacing condition and certain restrictions on the location of the poles of f.

We prove in Chapter 5 a result (Theorem 72) for this problem for any fixed $k \geq 1$ and a point set E, for the family of transcendental meromorphic functions with $\delta(\infty, f) > 1-1/k$. This condition allows us to remove the restriction on the position of the poles in Langley's result (ii), and to relax the spacing condition on the points of E.

1.4 Acknowledgements

I would like to thank my supervisor, Jim Langley, for his encouragement and expert guidance throughout the course of this work.

Chapter 2

Picard sets for functions with deficient poles

2.1 Introduction

We consider the family $\mathcal{F}$ of transcendental meromorphic functions f which have Nevanlinna deficient poles, ie

$$\delta(\infty, f) > 0.$$

Toppila [26] has proved the following:

Theorem 3 *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of complex numbers with*

$$\lim_{m\to\infty} a_m = \infty, \quad |a_1| > e$$

and, for some $0 < \alpha < 1$,

$$|a_m - a_{m'}| > \frac{|a_m|}{(\log |a_m|)^{\alpha}} \tag{2.1}$$

for all $m \neq m'$. If radii d_m are given by

$$\log 1/d_m = (\log |a_m|)^{2+\beta} \tag{2.2}$$

for some $\beta > 2\alpha$, *then the set*

$$S = \cup_{m=1}^{\infty} B(a_m, d_m),$$

where $B(a_m, d_m) = \{z : |z - a_m| < d_m\}$, *is a Picard set for* $\mathcal{F}$.

Toppila also showed [25] that this result is best possible, in the sense that if $\beta < 2\alpha$, there exists S satisfying (2.1) and (2.2) which is not even a Picard set for the family of entire functions.

Theorem 3 shows that, for any given $f \in \mathcal{F}$, the pre-image of at most two values of $\mathbf{C}^*$ may be contained in the set S.
The fact that any such f has deficient poles means that at most one finite value may be omitted (except for finitely often) in the whole plane, by Nevanlinna's second fundamental theorem.
This suggests the question, if two such exceptional values exist for a given $f \in \mathcal{F}$, whether they may both be finite or whether one of those values must be the deficient value ∞.

In this direction, Langley [18] has proved:

Theorem 4 *Let* $\{a_m\}_{m=1}^{\infty}$ *be a sequence of complex numbers with*

$$\lim_{m \to \infty} a_m = \infty$$

and

$$|a_m - a_{m'}| > \varepsilon |a_m|, \quad m \neq m' \tag{2.3}$$

for some $0 < \varepsilon < 1/2$. *Then there exists* $K = K(\varepsilon) > 0$ *such that, if radii* d_m *are given by*

$$\log 1/d_m > K(\log |a_m|)^2 \tag{2.4}$$

and

$$S = \cup_{m=1}^{\infty} B(a_m, d_m),$$

then every $f \in \mathcal{F}$ takes every finite value, with at most one exception, infinitely often in $\mathbf{C} - S$.

Note that spacing condition (2.3) on the points a_m is more restrictive in Langley's result than in Toppila's (2.1). We aim to go some way towards closing the gap between these two results. We shall prove the following:

Theorem 5 *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of complex numbers with*

$$\lim_{m \to \infty} a_m = \infty$$

and, for some $0 < \alpha < \frac{1}{4}$,

$$|a_m - a_{m'}| > \frac{|a_m|}{(\log |a_m|)^{\alpha}} \tag{2.5}$$

for all $m \neq m'$. Let radii $d_m = d(|a_m|)$ be given by

$$\log 1/d(r) = (\log r)^{2+\beta} \tag{2.6}$$

for some $\beta > 4\alpha$, and set

$$S = \cup_{m=1}^{\infty} B(a_m, d_m).$$

Then every $f \in \mathcal{F}$ takes every finite value, with at most one exception, infinitely often in $\mathbf{C} - S$.

Note that the tighter constraints on α and β mean that we have not closed the gap completely between Langley's result and Toppila's. It remains open as to whether both the omitted values in Toppila's result may be finite, when $\frac{1}{4} \leq \alpha < 1$ or when $2\alpha < \beta \leq 4\alpha$.

Our restrictions on α and β may be relaxed when the a_m lie on a ray. We prove:

Theorem 6 *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of positive real numbers with*

$$\lim_{m\to\infty} a_m = \infty$$

and, for some $0 < \alpha < \frac{1}{2}$,

$$|a_m - a_{m'}| > \frac{a_m}{(\log a_m)^\alpha} \tag{2.7}$$

for all $m \neq m'$. Let radii $d_m = d(a_m)$ be given by

$$\log 1/d(r) = (\log r)^{2+\beta} \tag{2.8}$$

for some $\beta > 2\alpha$, and set

$$S = \cup_{m=1}^{\infty} B(a_m, d_m).$$

Then every $f \in \mathcal{F}$ takes every finite value, with at most one exception, infinitely often in $\mathbf{C} - S$.

2.2 Lemmas needed for the proof of Theorem 5

The following is a modification by Langley [18] of an argument of Toppila.

Lemma 7 *Let $0 < t < s < r$ and assume that*

$$s_j > 0, \quad t < |b_j| - s_j < |b_j| + s_j < s \tag{2.9}$$

for $j = 1, \ldots, M$. Set

$$\Omega = \{z : t < |z| < r\} - \cup_{j=1}^{M} E_j, \tag{2.10}$$

where E_j is the closed disc $\{z : |z - b_j| \leq s_j\}$. Let u be subharmonic and non-positive on Ω, and continuous on the closure of Ω, and let $v(z)$ be the Poisson integral

$$v(z) = \frac{1}{2\pi} \int_0^{2\pi} -u(re^{i\theta}) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta \tag{2.11}$$

of $-u$ in $B(0, r)$. Then for z in Ω we have

$$u(z) \leq -v(z) + C(z)m_0(r, -u) \leq \left(\frac{|z| - r}{|z| + r} + C(z) \right) m_0(r, -u), \tag{2.12}$$

in which

$$m_0(r, -u) = \frac{1}{2\pi} \int_0^{2\pi} -u(re^{i\theta}) d\theta \tag{2.13}$$

and

$$C(z) = \frac{1 + t/r}{1 - t/r} \frac{\log(r/|z|)}{\log(r/t)} + \frac{1 + s/r}{1 - s/r} \sum_{j=1}^{M} \frac{\log(2r/|z - b_j|)}{\log(2r/s_j)}. \tag{2.14}$$

We recall the definitions of harmonic and subharmonic functions.

A function $v : \mathbf{C} \to \mathbf{R}$ is harmonic if it has continuous second partial derivatives which satisfy the Laplace equation

$$v_{xx} + v_{yy} = 0.$$

In particular, a function which is the real part of an analytic function is harmonic (consider the Cauchy-Riemann equations). For instance, the function v in Lemma 7 is harmonic, because

$$\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} = Re\left(\frac{re^{i\theta} + z}{re^{i\theta} - z}\right). \tag{2.15}$$

A continuous function $u : \mathbf{C} \to \mathbf{R}$ is subharmonic if for each z_0 there exists $r_0 > 0$ such that

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt, \quad 0 < r < r_0. \tag{2.16}$$

A harmonic function is also subharmonic, with equality in (2.16).

We shall also require (eg [8, p1]):

Theorem 8 (Poisson-Jensen formulae) *Suppose that g is meromorphic in $|z| \leq R$ and that g has zeros $\{x_\mu\}_{\mu=1}^X$ and poles $\{y_\nu\}_{\nu=1}^Y$ in $|z| < R$. Then if $z = re^{i\theta}$ and $g(0) \neq 0, \infty$, we have (*the Poisson-Jensen formula*)*

$$\log|g(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\phi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$
$$+ \sum_{\mu=1}^X \log\left|\frac{R(z - x_\mu)}{R^2 - \bar{x}_\mu z}\right| - \sum_{\nu=1}^Y \log\left|\frac{R(z - y_\nu)}{R^2 - \bar{y}_\nu z}\right|.$$

*Further, we have (*the differentiated Poisson-Jensen formula*)*

$$\begin{aligned}\frac{g'(z)}{g(z)} &= \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\phi})| \frac{2Re^{i\phi}}{(Re^{i\phi} - z)^2} d\phi \\ &\quad - \sum_{\mu=1}^X \left(\frac{1}{x_\mu - z} - \frac{\bar{x}_\mu}{R^2 - \bar{x}_\mu z}\right) + \sum_{\nu=1}^Y \left(\frac{1}{y_\nu - z} - \frac{\bar{y}_\nu}{R^2 - \bar{y}_\nu z}\right)\end{aligned}$$

provided there are no zeros or poles of g on $|z| = R$.

A useful inequality coming out of the differentiated Poisson-Jensen formula is (see eg [12, p65]):

Corollary 9 *Suppose that g is meromorphic in $|z| \leq R$. Then for $|z| = r < R$ we have*

$$\left|\frac{g'(z)}{g(z)}\right| \leq (m(R,g) + m(R,1/g))\frac{2R}{(R-r)^2} + 2\sum \frac{1}{|z - \zeta|} + \frac{d}{r}, \qquad (2.17)$$

where the sum is over all zeros and poles ζ of g with $|\zeta| < R$, repeated according to multiplicity, and d is the multiplicity of the zero or pole of g at $z = 0$, if any.

We shall also use the following elementary Nevanlinna theory result (see eg [8, p5]):

Lemma 10 *Let h be a non-constant meromorphic function, and let a be a complex number. Then*

$$|T(r,h) - T(r,h-a)| \leq \log^+ |a| + \log 2. \tag{2.18}$$

We also need the following, from [8, p38] and attributed to Borel:

Lemma 11 *Suppose that $T(r) \geq 1$ is a continuous and increasing function. Then*

$$T(r + 1/T(r)) < 2T(r) \tag{2.19}$$

outside a set of r of linear measure at most 2.

Also, we shall need:

Theorem 12 (Argument Principle) *Let f be a meromorphic function. Let C be a simple closed curve such that f does not take the values 0, ∞ on C. Let X be the domain enclosed by C. If $N(f)$ denotes the number of zeros of f in X, and $P(f)$ denotes the number of poles of f in X, in both cases counting multiplicities, we have*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} = N(f) - P(f).$$

2.3 Proof of Theorem 5

We follow Langley's method [18].

Let the a_m, d_m and α, β be as in the statement of the theorem. Suppose that there exists a transcendental meromorphic f which satisfies

$$\delta = \delta(\infty, f) > 0 \tag{2.20}$$

and which has all but finitely many of its zeros and 1-points in $S = \cup_{m=1}^{\infty} B(a_m, d_m)$. [This does not give rise to any loss of generality. If f omits distinct finite values a and $b \neq 0$, we may consider the function $\frac{f-a}{b}$, which omits 0 and 1.]

We shall show that these assumptions lead to a contradiction. This will prove the theorem.

We set

$$g = \frac{f-1}{f} = 1 - 1/f \tag{2.21}$$

so that, using Lemma 10 and the first fundamental theorem,

$$T(r, g) = T(r, 1/f) + O(1) = T(r, f) + O(1),$$

and also

$$\delta(1, g) = 1 - \limsup_{r\to\infty} \frac{N(r, 1, g)}{T(r, g)} = 1 - \limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)} = \delta \tag{2.22}$$

and all but finitely many of g's poles and zeros lie in S.

Throughout the proof, C_j will denote positive constants.

The strategy of the proof will be as follows:

- We find a sequence of annuli $B_n \to \infty$ which are away from the a_m and which satisfy certain minimum width conditions (Lemma 13);
- Inside each such annulus, we position three circles (centre 0) not too close together (Lemma 14);
- We obtain an upper bound for $|g'/g|$ inside each disc $|z| < S_n$ (Lemma 15), (where S_n is the radius of the middle circle);
- We use Lemma 7 and the deficiency condition to show that $\log|g'/g|$ is negative away from the a_m, for large enough z (2.55);
- It will follow that g is close to 1 (and hence f is large) away from the a_m, for large enough z (Lemma 17);
- We show that, for large m, g has the same number of zeros and poles in each $B(a_m, d_m)$ (Lemma 18);
- We show that g is still close to 1 (and hence f large) much nearer to (though still away from) the a_m (Lemma 20);
- These last two facts together will imply that f has at least as many poles as zeros near each a_m, for m large (Lemma 22);
- This will give a contradiction to $\delta(\infty, f) > 0$, and complete the proof.

We begin with:

Lemma 13 *Choose constants k, l with*

$$e < k < l < e^{9/8}.$$

There exists a constant $c > 0$ and a sequence $r_n \to \infty$ with

$$kr_n < r_{n+1} < lr_n \tag{2.23}$$

such that, for each $m, n \in \mathbf{N}$,

$$B\left(a_m, \frac{c|a_m|}{(\log|a_m|)^{2\alpha}}\right) \cap B_n = \emptyset, \tag{2.24}$$

where

$$B_n = \{z : r_n - \frac{cr_n}{(\log r_n)^{2\alpha}} < |z| < r_n + \frac{cr_n}{(\log r_n)^{2\alpha}}\}. \tag{2.25}$$

We construct the sequence r_n inductively.

Since g has only finitely many zeros and poles outside S, we can choose r_0 such that all the zeros and poles of g outside S lie inside $\{|z| < r_0\}$. We now use r_0 to begin the inductive argument (without requiring that r_0 satisfy (2.24)).

Suppose we have obtained r_n. It suffices to show that we can construct r_{n+1} satisfying (2.23), (2.24) and (2.25).

We may assume that all the a_m are large, so that by (2.5) the discs $B(a_m, \frac{|a_m|}{3(\log|a_m|)^{\alpha}})$ are disjoint. Each has area $\frac{\pi|a_m|^2}{9(\log|a_m|)^{2\alpha}}$.

Let

$$A_n = \{z : kr_n < |z| < lr_n\}. \tag{2.26}$$

The area of A_n is $\pi(l^2 - k^2)r_n^2$.

We therefore see that A_n contains at most

$$\lambda(r_n) = C_0(\log r_n)^{2\alpha} \tag{2.27}$$

of the a_m.

Hence we can find a sub-interval I_n of $[kr_n, lr_n]$, of length at least

$$\frac{(l-k)r_n}{\lambda(r_n)+1} \geq \frac{C_1 r_n}{(\log r_n)^{2\alpha}},$$

such that I_n contains none of the $|a_m|$.

Let $[G, H]$ be the middle third of I_n and set

$$r_{n+1} = \frac{G+H}{2}.$$

Then

$$H - G \geq \frac{C_1 r_n}{3(\log r_n)^{2\alpha}} \geq \frac{C_2 r_{n+1}}{(\log r_{n+1})^{2\alpha}}, \tag{2.28}$$

using (2.26).

But now, noting that

$$\begin{aligned} H - G &\geq \frac{C_2 r_{n+1}}{(\log r_{n+1})^{2\alpha}} \geq \frac{C_2 k r_n}{(\log k r_n)^{2\alpha}} \\ &\geq \frac{C_2 k}{l} \frac{l r_n}{(\log l r_n)^{2\alpha}} \geq \frac{C_2 k}{l} \frac{|a_m|}{(\log |a_m|)^{2\alpha}} \end{aligned}$$

for any $a_m \in A_n$, we can choose c with

$$0 < c < \frac{C_2 k}{2l} < \frac{C_2}{2}$$

and then (2.24) will hold, using (2.28).

So r_{n+1} satisfies (2.23), (2.24) and (2.25), and by induction we have constructed the sequence $\{r_n\}$ as desired. Lemma 13 is proved.

Lemma 14 *Let γ, ε be positive constants with ε/c and γ/ε small, where c is as in Lemma 13. Then for each large n there exist S_n, S_n' with*

$$(1 + \frac{\gamma}{(\log r_n)^{2\alpha}})r_n < S_n < (1 + \frac{\varepsilon}{(\log r_n)^{2\alpha}})r_n \tag{2.29}$$

and

$$S_n' = S_n + 1/T(S_n, g) < r_n + \frac{cr_n}{(\log r_n)^{2\alpha}} \tag{2.30}$$

such that

$$T(S_n', g) < 2T(S_n, g) \tag{2.31}$$

$$m(S_n, g'/g(g-1)) < C_3 \log(S_n T(S_n, g)) \tag{2.32}$$

With S_n' defined as in (2.30), we have that (2.31) holds, by Lemma 11 (noting that the exceptional set in that result is of finite linear measure and therefore too small to cover all the intervals in (2.29), so that we can always choose suitable S_n and S_n', for large enough n).

For (2.32), we apply the lemma of the logarithmic derivative (1.17) to

$$\frac{g'}{g(g-1)} = \frac{g'}{g-1} - \frac{g'}{g}.$$

The inequality in (2.30) holds, from (2.29) and the fact that

$$T(S_n, g)^{-1} < (\log S_n)^{-1} < 2(\log r_n)^{-1} = o\left(\frac{r_n}{(\log r_n)^{2\alpha}}\right)$$

for large n. This inequality shows that the circle $|z| = S_n'$ lies inside the annulus B_n.

Lemma 14 is proved.

Lemma 15 *There exist positive constants C_6, C_7 such that, for large n, we have*

$$|g'(z)/g(z)| \leq C_6 S_n T(S_n, g)^3 \tag{2.33}$$

for all z satisfying

$$C_7 \leq |z| \leq S_n, \quad z \notin \cup_{m=1}^{\infty} B(a_m, 1). \tag{2.34}$$

We fix a large integer L and assume that n is large compared to L. We have, using (2.29),

$$\begin{aligned} T(S_n, g) &\geq N(r_n, g) + \int_{r_n}^{S_n} \frac{n(t,g)}{t} dt \\ &\geq N(r_n, g) + n(r_n, g) \log(S_n/r_n) \\ &\geq N(r_n, g) + n(r_n, g) \log\left(1 + \frac{\gamma}{(\log r_n)^{2\alpha}}\right) \\ &\geq \frac{\gamma}{2(\log r_n)^{2\alpha}} n(r_n, g) \end{aligned} \tag{2.35}$$

and so, since by construction there are no poles of g in $r_n < |z| < S_n'$,

$$n(S_n', g) = n(r_n, g) < \frac{2}{\gamma}(\log r_n)^{2\alpha} T(S_n, g).$$

Arguing similarly for $n(S_n', 1/g)$, we obtain

$$n(S_n', g) + n(S_n', 1/g) < C_4 (\log r_n)^{2\alpha} T(S_n, g). \tag{2.36}$$

Now we apply the differentiated Poisson-Jensen formula (Corollary 9) in $B(0, S_n')$ to get, for $|z| \leq S_n$,

$$|g'(z)/g(z)| \leq [m(S_n', g) + m(S_n', 1/g)] \frac{2S_n'}{(S_n' - S_n)^2} + 2\sum \frac{1}{|z - \zeta|} + C_5 \tag{2.37}$$

where the sum is taken over all zeros and poles ζ of g with $|\zeta| < S_n'$.

Now from (2.30),

$$\frac{2S'_n}{(S'_n - S_n)^2} < 4S_nT(S_n, g)^2 \tag{2.38}$$

and so we get, by (2.31), (2.36), (2.37) and (2.38),

$$\begin{aligned} |g'(z)/g(z)| &\leq 32S_nT(S_n, g)^3 + 2[n(S'_n, g) + n(S'_n, 1/g)] + C_5 \\ &\leq 32S_nT(S_n, g)^3 + 2C_4(\log r_n)^{2\alpha}T(S_n, g) + C_5 \\ &\leq C_6S_nT(S_n, g)^3 \end{aligned} \tag{2.39}$$

as long as $C_7 \leq |z| \leq S_n$ and $|z - a_m| \geq 1$ for all m, using the fact (1.15) that

$$\frac{T(r, g)}{\log r} \to \infty \tag{2.40}$$

as $r \to \infty$ since g is transcendental. Lemma 15 is proved.

Lemma 16 *For large enough n, we have*

$$m(S_n, g/g') > (\delta/2)T(S_n, g). \tag{2.41}$$

We have

$$\frac{1}{g-1} = \frac{g'}{g(g-1)}\frac{g}{g'}. \tag{2.42}$$

Now (2.22), (2.32) and (2.40) give

$$m(S_n, g/g') > (\delta/2)T(S_n, g) \tag{2.43}$$

for large enough n. Lemma 16 is proved.

Lemma 17 *Let* ε_1 *be small and positive, in particular with* $\varepsilon_1 < c/2$*, where* c *is as in Lemma 13. Then there exists* $C_{15} > 0$ *such that, for all large* n*, we have*

$$\log |g(z) - 1| < -C_{15}\delta T(r_n, g), \tag{2.44}$$

for all z *satisfying*

$$r_{n-1} \leq |z| \leq r_n, \quad z \notin \cup_{m=1}^{\infty} B\left(a_m, \frac{\varepsilon_1 |a_m|}{(\log |a_m|)^{2\alpha}}\right). \tag{2.45}$$

We apply Lemma 7 to the function

$$u(z) = \log |g'(z)/g(z)| - \log[C_6 S_{n+1} T(S_{n+1}, g)^3] \tag{2.46}$$

with $r = S_{n+1}$ and $t = r_{n-L}$ and $s = r_{n+1}$ and with the $B(b_j, s_j)$ those discs $B(a_m, 1)$ for which $t < |a_m| < r$. (Recall that L is a fixed large integer and that n is large compared to L.)

From (2.33) for $n+1$, u is non-positive in $t < |z| < r$ outside the $B(a_m, 1)$. u is the real part of an analytic function, and so harmonic and therefore subharmonic, from the remarks following Lemma 7.

Note that $r_{n-L} < |a_m| - 1 < |a_m| + 1 < r_{n+1}$ for all these m, by (2.24), so the choices of $t = r_{n-L}$, $s = r_{n+1}$ in Lemma 7 are permissible.

We take z satisfying

$$r_{n-1} \leq |z| \leq r_n, \quad |z - a_m| \geq \frac{\varepsilon_1 |a_m|}{(\log |a_m|)^{2\alpha}} \tag{2.47}$$

for all m. The fact that $\varepsilon_1 < c/2$ and (2.24) together ensure that any disc $B(a_m, \frac{\varepsilon_1 |a_m|}{(\log |a_m|)^{2\alpha}})$ which meets the annulus $r_{n-1} \leq |z| \leq r_n$ actually lies wholly inside that annulus.

We then have

$$\begin{aligned}\frac{|z|-r}{|z|+r} = \frac{|z|/r-1}{|z|/r+1} &\leq \frac{r_n/S_{n+1}-1}{r_n/S_{n+1}+1} \\ &\leq \frac{r_n/r_{n+1}-1}{r_n/r_{n+1}+1} \leq \frac{1/k-1}{1/k+1} \\ &= \frac{-(k-1)}{k+1} \equiv -\tau < 0 \end{aligned} \tag{2.48}$$

since the function $\frac{x-1}{x+1}$ is increasing on $(0, 1)$.

Further, using (2.29),

$$\frac{1+s/r}{1-s/r} = 1 + \frac{2s/r}{1-s/r} \leq 1 + \frac{2}{\gamma}(\log r_{n+1})^{2\alpha} < \frac{3}{\gamma}(\log r_{n+1})^{2\alpha}. \tag{2.49}$$

Also, $r_{n+1} \leq l^{L+1} r_{n-L}$ and so the number of a_m between $|z| = r_{n-L}$ and $|z| = r_{n+1}$ is at most

$$9(l^{2L+2}-1)(\log r_{n-L})^{2\alpha} \tag{2.50}$$

(by the same argument used to obtain (2.27), by comparing the area of the annulus to the combined area of non-intersecting discs around a_m which meet the annulus).

For each a_m with $r_{n-1} < |a_m| < r_n$, we have, using (2.47),

$$\begin{aligned}\frac{\log(2S_{n+1}/|z-a_m|)}{\log(2S_{n+1})} &\leq \frac{\log[\frac{2S_{n+1}(\log|a_m|)^{2\alpha}}{\varepsilon_1|a_m|}]}{\log 2S_{n+1}} \\ &\leq \frac{\log[\frac{2lr_n(\log r_{n-1})^{2\alpha}}{\varepsilon_1 r_{n-1}}]}{\log 2S_{n+1}} \\ &\leq \frac{C_8 + 2\alpha\log\log r_{n-1}}{\log 2S_{n+1}}. \end{aligned} \tag{2.51}$$

Also,

$$\frac{1 + r_{n-L}/S_{n+1}}{1 - r_{n-L}/S_{n+1}} \frac{\log(S_{n+1}/|z|)}{\log(S_{n+1}/r_{n-L})} < 2\frac{\log(S_{n+1}/r_{n-1})}{\log(k^{L+1})} \leq \frac{6\log l}{L\log k}. \tag{2.52}$$

Therefore, in (2.14) we have, using (2.49), (2.50), (2.51) and (2.52),

$$\begin{aligned} 0 < C(z) &= \frac{1 + r_{n-L}/S_{n+1}}{1 - r_{n-L}/S_{n+1}} \frac{\log(S_{n+1}/|z|)}{\log(S_{n+1}/r_{n-L})} + \frac{1+s/r}{1-s/r} \sum_{r_{n-L}<|a_m|<S_{n+1}} \frac{\log(2S_{n+1}/|z-a_m|)}{\log(2S_{n+1})} \\ &\leq \frac{6\log l}{L\log k} + \frac{3}{\gamma}(\log r_{n+1})^{2\alpha} 9 l^{2L+2} (\log r_{n-L})^{2\alpha} \frac{C_8 + 2\alpha \log\log r_{n-1}}{\log(2S_{n+1})} \\ &\leq \frac{6\log l}{L\log k} + \frac{C_9 l^{2L+2} \log\log r_{n-1}}{(\log S_{n+1})^{1-4\alpha}} \\ &< \tau/2 \end{aligned} \tag{2.53}$$

for large enough L, n since $\alpha < 1/4$.

Now (2.48) and (2.53) give that

$$\frac{|z| - r}{|z| + r} + C(z) < -\tau/2. \tag{2.54}$$

Also, we have from (2.33) and (2.46),

$$-u(z) = \log|g(z)/g'(z)| + \log(C_6 S_{n+1} T(S_{n+1}, g)^3) \geq \log^+ |g(z)/g'(z)|,$$

since either $\log^+ |g(z)/g'(z)| = \log|g(z)/g'(z)| \geq 0$, in which case the inequality clearly holds, or else the right hand side is zero, in which case the inequality still holds, by (2.33).

So, integrating, we obtain

$$m_0(r, -u) \geq m(r, g/g')$$

and so this, together with (2.12), (2.46) and (2.54) give

$$\log|g'(z)/g(z)| - \log(C_6 S_{n+1} T(S_{n+1}, g)^3) \leq -(\tau/2) m(S_{n+1}, g/g')$$

and so, using (1.15) and (2.43),

$$\begin{aligned}\log|g'(z)/g(z)| &\leq (-\tau/2)m(S_{n+1}, g/g') + \log(C_6 S_{n+1} T(S_{n+1}, g)^3) \\ &\leq -C_{10}\delta T(S_{n+1}, g) \\ &\leq -C_{10}\delta T(r_n, g) \qquad (2.55)\end{aligned}$$

for n large enough and z as in (2.47).

Now, we choose z_0 satisfying $|z_0| = r_n$ such that $g(z_0)$ is so close to 1 that

$$|\log g(z_0)| \leq \exp(-\frac{1}{2}\delta T(r_n, g)).$$

This is possible, since otherwise

$$|g(z) - 1| \geq \frac{1}{2}|\log g(z)| > \exp(-\frac{1}{2}\delta T(r_n, g))$$

for $|z| = r_n$ and then

$$m(r_n, 1/(g-1)) \leq \frac{1}{2}\delta T(r_n, g)$$

which contradicts (2.22).

For any z as in (2.47) we have

$$\begin{aligned}|\log g(z)| &= \left|\log g(z_0) + \int_{z_0}^{z} g'(w)/g(w)dw\right| \\ &\leq \exp(-\frac{1}{2}\delta T(r_n, g)) + L(z, z_0)\exp[-C_{10}\delta T(r_n, g)]\end{aligned}$$

by (2.55), where $L(z, z_0)$ is the length of a path between z_0 and z which is entirely within the region given in (2.47). By considering a path from z_0 to z which consists of a ray from z_0 to $|z/z_0|z_0$ followed by an arc from this point to z, and which is "diverted" round the boundary of any disc $B(a_m, \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}})$ which it meets, we see, using (2.27), that $L(z, z_0)$ need be no greater than

$$(l-1)r_{n-1} + \pi l r_{n-1} + \pi \frac{\varepsilon_1 |a_m|}{(\log |a_m|)^{2\alpha}} C_{11} (\log(r_n))^{2\alpha} \leq C_{12} r_n$$

and so we obtain, using (2.40),

$$\begin{aligned} |\log g(z)| &\leq \exp(-\frac{1}{2}\delta T(r_n, g)) + C_{13} r_n \exp[-C_{10} \delta T(r_n, g)] \\ &\leq \exp[-C_{14} \delta T(r_n, g)]. \end{aligned}$$

So $g(z)$ is very close to 1 for large n and for z satisfying (2.47), and so

$$|g(z) - 1| \leq \exp\exp[-C_{14} \delta T(r_n, g)] - 1 < 2 \exp[-C_{14} \delta T(r_n, g)]$$

and so

$$\log |g(z) - 1| < -C_{15} \delta T(r_n, g) \tag{2.56}$$

for n large enough and z as in (2.47). Lemma 17 is proved.

Lemma 18 *g has the same number of zeros as poles inside each $B(a_m, d_m)$, for large enough m.*

We observe from (2.56) that $g(z)$ stays close to 1 as z traverses the circle $C = \{|z - a_m| = \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}}\}$. So $\log g(z)$ returns to its original value and we have

$$0 = \frac{1}{2\pi i} [\log g(z)] = \frac{1}{2\pi i} \int_C g'(w)/g(w) dw.$$

We conclude, using the Argument Principle (Theorem 12), that g has the same number of zeros as poles inside each $B(a_m, \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}})$, for large enough m. But all of g's large zeros and poles are inside the smaller discs $B(a_m, d_m)$, and so Lemma 18 is proved.

Lemma 19 *There exists $\rho > 0$ such that*

$$T(r, g) < r^{\rho} \tag{2.57}$$

for all large enough r.

For large enough n,

$$T(r_n, g) \leq \frac{1}{C_{15}\delta} m(r_{n-1}, 1/(g-1)) \leq C_{16}\delta^{-1} T(r_{n-1}, g) = c_1 T(r_{n-1}, g)$$

by (2.56). So, for some large fixed J and n large compared to J,

$$T(r_n, g) \leq c_1^{n-J} T(r_J, g) \leq c_2 c_1^n,$$

for some constant $c_2 > 0$. Now,

$$r_n > k r_{n-1} > \ldots > k^n r_0$$

so

$$n < c_3 \log r_n$$

for a constant c_3, and so

$$T(r_n, g) \leq c_2 c_1^{c_3 \log r_n} = c_2 r_n^{c_3 \log c_1} < r_n^{\rho_1}$$

for some $\rho_1 > 0$.

But now, given any large r, we find n such that $r_{n-1} \leq r < r_n$ and then

$$T(r, g) \leq T(r_n, g) \leq r_n^{\rho_1} \leq (l r_{n-1})^{\rho_1} \leq l^{\rho_1} r^{\rho_1} < r^{\rho},$$

choosing ρ so that $r_L^{\rho-\rho_1} > l^{\rho_1}$.

Lemma 19 is proved.

Lemma 20 *We have $g(z) = 1 + o(1)$ as $z \to \infty$ outside the union of the discs $B(a_m, \sqrt{d_m})$.*

We have already shown at (2.56) that this result holds outside the discs $B(a_m, \frac{\varepsilon_1 |a_m|}{(\log |a_m|)^{2\alpha}})$, so it suffices to prove that $g(z) = 1 + o(1)$ for

$$\sqrt{d_m} \leq |z - a_m| \leq \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}}. \tag{2.58}$$

Fix m large. From Lemma 18, g has the same number (N say) of zeros and poles in $B(a_m, d_m)$. We pair off the zeros ζ_ν and poles ξ_ν of g in $B(a_m, d_m)$, $1 \leq \nu \leq N$, and set

$$U(z) = g(z)P(z), \quad P(z) = \prod_{\nu=1}^{N} \frac{z - \xi_\nu}{z - \zeta_\nu}$$

so that U is analytic, non-zero in $B(a_m, \frac{\varepsilon_1 |a_m|}{(\log |a_m|)^{2\alpha}})$. For z satisfying (2.58), we have

$$\begin{aligned} |\log P(z)| &= \left| \sum_{\nu=1}^{N} \log \frac{z - \xi_\nu}{z - \zeta_\nu} \right| = \left| \sum_{\nu=1}^{N} \log(1 + \frac{\zeta_\nu - \xi_\nu}{z - \zeta_\nu}) \right| \\ &\leq 2 \sum_{\nu=1}^{N} \left| \frac{\zeta_\nu - \xi_\nu}{z - \zeta_\nu} \right| < 2N \frac{2d_m}{\sqrt{d_m} - d_m} < 8N\sqrt{d_m}. \end{aligned} \tag{2.59}$$

If $|z - a_m| = \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}}$, then

$$|\log U(z)| \leq |\log g(z)| + |\log P(z)| < o(1) + 8N\sqrt{d_m} = o(1) \tag{2.60}$$

as $m \to \infty$ since

$$N\sqrt{d_m} < O(|a_m|^{\rho - \frac{1}{2}(\log |a_m|)^{1+\beta}}) = o(1),$$

using (2.6) and (2.57). So, by the maximum principle, $\log U(z) = o(1)$ inside $B(a_m, \frac{\varepsilon_1 |a_m|}{(\log a_m)^{2\alpha}})$ also. But now, for z satisfying (2.58), we get

$$\log g(z) = o(1)$$

from (2.59) and (2.60), and Lemma 20 follows. This shows in particular that $|f(z)|$ is large for large z outside the $B(a_m, \sqrt{d_m})$.

Next we prove:

Lemma 21 *$T(r, f) = o((\log r)^2)$ as $r \to \infty$.*

We take n large and apply Lemma 7 with

$$r = r_n, \quad t = r_{n'}, \quad r^{1/100} < t < r^{1/70}, \quad s = r - \frac{cr}{2(\log r)^{2\alpha}}, \quad u(z) = -\log |f(z)|$$

where c is the constant in Lemma 13, and with the $B(b_j, s_j)$ those $B(a_m, \sqrt{d_m})$ for which $t < |a_m| < r$.

Then

$$\frac{1 + s/r}{1 - s/r} < C_{17} (\log r)^{2\alpha}. \tag{2.61}$$

We have that $u(z) \leq 0$ for z outside the $B(b_j, s_j)$ since f is large there, by Lemma 20.

For $|z| = r_{n-1}$, we have

$$\frac{1 + t/r}{1 - t/r} \frac{\log r/|z|}{\log r/t} < \frac{8}{7 \log r} \tag{2.62}$$

using the fact that $r_n / r_{n-1} < e^{9/8}$, and

$$\log \frac{2r}{|z - b_j|} \leq C_{16} \alpha \log \log r \tag{2.63}$$

using (2.24) and (2.25) for r_{n-1}. Also,

$$\log\frac{2r}{s_j}\geq\log\frac{2r}{\sqrt{d}(t)} \tag{2.64}$$

for every j, since $|b_j|\geq t$.

We calculate the maximum number of a_m in the annulus $t<|z|<r$. We note that

$$\int_{t\leq|z|\leq r}\frac{1}{|z|^2}dxdy=2\pi\log(r/t)\leq C_{18}\log r$$

and, for any m with $t<|a_m|<r$,

$$\begin{aligned}\int_{B(a_m,\frac{|a_m|}{3(\log|a_m|)^\alpha})}\frac{1}{|z|^2}dxdy &\geq \frac{\pi|a_m|^2}{9(\log|a_m|)^{2\alpha}}\frac{1}{4|a_m|^2}\\ &\geq \frac{C_{19}}{(\log|a_m|)^{2\alpha}}\geq\frac{C_{19}}{(\log r)^{2\alpha}}\end{aligned}$$

and so, since the discs $B(a_m,\frac{|a_m|}{3(\log|a_m|)^\alpha})$ are disjoint, the maximum number of a_m in the annulus $t<|z|<r$ is no more than

$$C_{20}(\log r)^{1+2\alpha}. \tag{2.65}$$

So using (2.6), (2.61), (2.62), (2.63), (2.64) and (2.65), Lemma 7 gives, for $|z|=r_{n-1}$,

$$\begin{aligned}-\log|f(z)|=u(z) &\leq -v(z)+m(r,f)[\frac{8}{7\log r}+C_{21}(\log r)^{1+4\alpha}\frac{\alpha\log\log r}{\log\frac{2r}{\sqrt{d}(t)}}]\\ &\leq -v(z)+m(r,f)[\frac{8}{7\log r}+C_{22}(\log r)^{1+4\alpha}\frac{\alpha\log\log r}{(\log t)^{2+\beta}}]\\ &\leq -v(z)+m(r,f)\frac{7}{6\log r}\end{aligned}$$

for large enough n, using the fact that $\beta>4\alpha$.

But v is harmonic in $B(0,r)$, for the reasons given at (2.15), with $v(0) = m_0(r_n, -u) = m(r_n, f)$, and so integrating round $|z| = r_{n-1}$ we obtain

$$-m(r_{n-1}, f) \leq m(r_n, f)[-1 + \frac{7}{6 \log r_n}]$$

and therefore

$$\frac{m(r_n, f)}{m(r_{n-1}, f)} \leq 1 + \frac{6}{5 \log r_n} \leq 1 + \frac{5}{4n}$$

for large enough n, and so

$$\log m(r_n, f) \leq O(1) + \frac{3}{2} \log n \leq O(1) + \frac{3}{2} \log \log r_n$$

using the fact that (2.23) implies that $n < \log r_n < \frac{9}{8} n$ for large enough n. So then, by (2.20),

$$T(r_n, f) \leq \frac{2}{\delta} m(r_n, f) \leq \frac{2}{\delta} C_{23} (\log r_n)^{3/2} = o((\log r_n)^2).$$

Now, for any large r, we have $r_{n-1} < r \leq r_n$ for some n, and then

$$T(r, f) \leq T(r_n, f) = o((\log r_n)^2) = o((\log r_{n-1})^2) = o((\log r)^2)$$

since $r_n \leq l r_{n-1}$. Lemma 21 is proved.

In particular, we note that Lemma 21 implies that

$$n(r, f) + n(r, 1/f) = o(\log r). \tag{2.66}$$

Lemma 22 *Let $0 < \sigma < \varepsilon_1$ and let m be large. Then f has at least as many poles as zeros, counting multiplicity, in $B(a_m, \frac{\sigma|a_m|}{(\log|a_m|)^{2\alpha}})$.*

Set

$$h(z) = f(z)\prod_{\mu=1}^{p}(z-z_\mu)^{-1}\prod_{\nu=1}^{q}(z-w_\nu), \tag{2.67}$$

where the z_μ, $1 \leq \mu \leq p$, are the zeros and the w_ν, $1 \leq \nu \leq q$, are the poles of f in $B(a_m, \frac{\sigma|a_m|}{(\log|a_m|)^{2\alpha}})$. Then h is analytic and non-zero in $B(a_m, \frac{\sigma|a_m|}{(\log|a_m|)^{2\alpha}})$. We have

$$\begin{aligned}
T(4|a_m|, h) &\leq T(4|a_m|, f) + T(4|a_m|, \prod_{\mu=1}^{p}(z-z_\mu)^{-1}) + T(4|a_m|, \prod_{\nu=1}^{q}(z-w_\nu)) \\
&\leq T(4|a_m|, f) + T(4|a_m|, \prod_{\mu=1}^{p}(z-z_\mu)) + T(4|a_m|, \prod_{\nu=1}^{q}(z-w_\nu)) \\
&\leq T(4|a_m|, f) + O(n(2|a_m|, f) + n(2|a_m|, 1/f))\log|a_m| = o((\log|a_m|)^2)
\end{aligned}$$

by Lemma 21 and (2.66).

We apply the Poisson-Jensen formula (Theorem 8) to h in $|\xi| < R = 2|a_m|$. Since

$$\left|\frac{R(z-a)}{R^2 - \bar{a}z}\right| \leq 1$$

for $|z|, |a| < R$, the contribution from zeros of h is non-positive and so we can ignore it.

This gives, for $z \in B(a_m, 1)$,

$$\begin{aligned}
\log|h(z)| &\leq \frac{2|a_m| + |a_m|}{2|a_m| - |a_m|}(1+o(1))(m(2|a_m|, h) + m(2|a_m|, 1/h)) \\
&\quad + n(2|a_m|, h)(1+o(1))\log\left|\frac{8|a_m|^2(\log|a_m|)^{2\alpha}}{2|a_m|\sigma|a_m|}\right| \\
&\leq 7T(2|a_m|, h) + n(2|a_m|, h)(2\alpha\log\log|a_m| + O(1)) \\
&= o((\log|a_m|)^2)
\end{aligned} \tag{2.68}$$

from the estimate above.

We choose z with $\sqrt{d_m} \le |z - a_m| \le 4\sqrt{d_m}$ and $|z - w_\nu| \ge \sqrt{d_m}/q$ for each ν. Then Lemma 20 and Lemma 21 and (2.67) and (2.68) give

$$\begin{aligned} 0 &\le \log|f(z)| \\ &\le \log|h(z)| + p\log|4\sqrt{d_m} + d_m| - q\log\sqrt{d_m}/q \\ &\le o((\log|a_m|)^2) + p\log 5 + q\log q + (p-q)\log\sqrt{d_m}. \end{aligned} \tag{2.69}$$

Now $-\log\sqrt{d_m} = \frac{1}{2}(\log|a_m|)^{2+\beta}$ and in particular $-\log\sqrt{d_m} \ne o((\log|a_m|)^2)$. Also,

$$q \le n(2|a_m|, f) \le \frac{N(2|a_m|^2, f)}{\log|a_m|} = o(\log|a_m|)$$

and so

$$q\log q = o((\log|a_m|)^2).$$

Similarly,

$$p \le n(2|a_m|, 1/f) = o(\log|a_m|)$$

and so

$$p\log 5 = o((\log|a_m|)^2).$$

Therefore the $(p-q)\log\sqrt{d_m}$ term dominates in (2.69) (unless $p = q$) and so we conclude that $p \le q$. Lemma 22 is proved.

Lemma 23 *For large n we have*

$$N(r_n, 1/f) \le (1 + o(1))N(r_n, f). \tag{2.70}$$

By Lemma 20 and the fact that g is transcendental, f has infinitely many zeros. If m is large and $|a_m| < r_n$, then Lemma 22 shows that to each zero z_μ

of f in $B(a_m, d_m)$ there corresponds a pole w_ν of f with

$$w_\nu = z_\mu(1 + o(1)), \quad \log\frac{r_n}{|z_\mu|} \le \log\frac{r_n}{|w_\nu|} + o(1).$$

This gives

$$N(r_n, 1/f) \le N(r_n, f) + O(\log r_n) + o(n(r_n, 1/f)) = N(r_n, f) + O(\log r_n)$$

and Lemma 23 follows.

But now we can complete the proof of the theorem. Since f is large on $|z| = r_n$, by Lemma 20, we have for large n,

$$T(r_n, f) = N(r_n, 1/f) + O(1) \le (1 + o(1))N(r_n, f) \le (1 - \delta/2)T(r_n, f)$$

which is a contradiction. Theorem 5 is proved.

2.4 Proof of Theorem 6

The proof proceeds as for Theorem 5, with the following modifications.

At (2.23) and (2.24), we construct the sequence $\{r_n\}_{n=1}^{\infty}$ such that

$$||a_m| - r_n| \ge \frac{cr_n}{(\log r_n)^\alpha}, \quad kr_n < r_{n+1} < lr_n. \tag{2.71}$$

We define the annulus A_n as before and note that it has width $(l - k)r_n$. We note that the intervals $[a_m - \frac{a_m}{3(\log a_m)^\alpha}, a_m + \frac{a_m}{3(\log a_m)^\alpha}]$ do not intersect and therefore A_n can meet at most

$$\bar{\lambda}(r_n) = C_{24}(\log r_n)^\alpha \tag{2.72}$$

of the $B(a_m, d_m)$.

We can therefore find a thinner annulus B_n inside A_n of width at least

$$H - G \geq \frac{C_{25} r_n}{(\log r_n)^\alpha}$$

which does not intersect any of the $B(a_m, d_m)$, and so we can define

$$r_{n+1} = (G + H)/2$$

as before.

We choose S_{n+1} and S'_{n+1} with obvious modifications.

At (2.47), we take

$$r_{n-1} \leq |z| \leq r_n, \quad |z - a_m| \geq \frac{\varepsilon_1 a_m}{(\log a_m)^\alpha} \tag{2.73}$$

for all m and some small fixed $\varepsilon > 0$.

(2.49) becomes

$$\frac{1 + s/r}{1 - s/r} < \frac{3}{\gamma}(\log r_{n+1})^\alpha. \tag{2.74}$$

The number of a_m between $|z| = r_{n-L}$ and $|z| = r_{n+1}$ is at most

$$(l^{L+1} - 1)(\log r_{n-L})^\alpha. \tag{2.75}$$

At (2.53) we have

$$\begin{aligned} 0 < C(z) &\leq \frac{C_7}{L} + \frac{3}{\gamma}(\log r_{n+1})^\alpha l^{L+1} (\log r_{n-L})^\alpha \frac{C_8 + \alpha \log\log r_{n-1}}{\log(2S_{n+1})} \\ &\leq \frac{C_7}{L} + \frac{C_9 l^{L+1} \log\log r_{n-1}}{(\log S_{n+1})^{1-2\alpha}} \\ &< \tau/2 \end{aligned} \tag{2.76}$$

for large enough L, n since $\alpha < 1/2$.

The proof proceeds, with obvious modifications (ie replacing $(\log |a_m|)^{2\alpha}$ with $(\log a_m)^\alpha$ where appropriate), until we reach Lemma 21. Here, we apply Lemma 7 as before except that we use

$$s = r - \frac{cr}{2(\log r)^\alpha}.$$

Then

$$\frac{1 + s/r}{1 - s/r} < C_{17}(\log r)^\alpha.$$

We calculate the maximum number of a_m between t and r. We have that

$$\int_t^r \frac{dx}{x} = \log(r/t) < C_{18} \log r.$$

We also have

$$\int_{a_m - \frac{a_m}{3(\log a_m)^\alpha}}^{a_m + \frac{a_m}{3(\log a_m)^\alpha}} \frac{dx}{x} > \frac{C_{19}}{(\log a_m)^\alpha}.$$

So the number of a_m between t and r is at most

$$C_{20}(\log r)^{1+\alpha}.$$

So we get

$$\begin{aligned} -\log |f(z)| &\leq -v(z) + m(r, f)[\frac{8}{7 \log r} + C_{21}(\log r)^{1+2\alpha} \frac{\alpha \log \log r}{(\log t)^{2+\beta}}] \\ &\leq -v(z) + m(r, f) \frac{7}{6 \log r} \qquad (2.77) \end{aligned}$$

for large enough n using the fact that $\beta > 2\alpha$.

Now the remainder of the proof proceeds as before. Theorem 6 is proved.

Chapter 3

Analogues of Picard sets for the problem $f^N f' = 1$

3.1 Introduction

Our starting point is the following theorem, which is due to Hayman for $N \geq 3$ [9], Mues for $N = 2$ [20] and Bergweiler/Eremenko for $N = 1$ [6]:

Theorem 24 *Let f be a transcendental meromorphic function in the complex plane* $\mathbf{C}$*, and let $N \geq 1$ be an integer. Then $f^N f'$ takes all finite non-zero complex values infinitely often.*

The question arises as to what sets E may be excluded from $\mathbf{C}$ such that, for any transcendental meromorphic f, we have that $f^N f'$ takes all finite non-zero values infinitely often on $\mathbf{C} - E$.

In this direction, Anderson, Baker and Clunie [2] proved the following two results for a point set E:

Theorem 25 *Let $q > 1$ and let $E = \{a_m\}$ be a set of complex numbers a_m such that*

$$\left|\frac{a_{m+1}}{a_m}\right| > q$$

for all m.

Then, for any non-zero complex b, for integer $N \geq 2$ and for f a transcendental entire function, the equation

$$f^N(z)f'(z) = b$$

has infinitely many solutions in $\mathbf{C} - E$.

Theorem 26 *Let $E = \{a_m\}$ be a set of complex numbers a_m such that $a_m \to \infty$ and*

$$\liminf_{m\to\infty} \frac{\log|a_{m+1}|}{\log|a_m| \log\log|a_m|} > 0.$$

Then, for any non-zero complex b, for integer $N \geq 11$ and for f a transcendental meromorphic function, the equation

$$f^N(z)f'(z) = b$$

has infinitely many solutions in $\mathbf{C} - E$.

Langley [14] proved the following result where E is a countable union of open discs:

Theorem 27 *Let $q > 1$ and let $\{a_m\}_{m=1}^{\infty}$ be a set of complex numbers and $\{d_m\}_{m=1}^{\infty}$ be a set of positive numbers such that, for all m,*

$$\left| \frac{a_{m+1}}{a_m} \right| > q$$

and

$$\log \frac{1}{d_m} > \frac{q^{1/4}+1}{q^{1/4}-1} \frac{8}{\log q} (\log |a_m|)^2.$$

Let b be any non-zero complex number. Then if f is a transcendental entire function, and $N \geq 2$ is an integer, the equation $f^N(z)f'(z) = b$ has infinitely many solutions outside the union of the discs

$$B(a_m, d_m) = \{z : |z - a_m| < d_m\}.$$

Our aim in this chapter will be as follows.

(i) We shall aim to bridge the gap, to some extent, between the two Anderson/Baker/Clunie results for a point set E, by considering the case when f is meromorphic but has deficient poles. We shall obtain a result (Theorem 28) which has the same spacing condition as Theorem 25.

(ii) We shall also aim to improve on Langley's result for discs by allowing f to be meromorphic with deficient poles. Again, we shall obtain a result (Theorem 29) which has the same spacing condition as Theorem 25 although, as one might expect, a stronger condition on the deficiency of f than in (i) is required.

For a point set E, we shall prove:

Theorem 28 *Let $q > 1$ and let $E = \{a_m\}$ be a set of complex numbers a_m such that*

$$|\frac{a_{m+1}}{a_m}| > q$$

for all m.

Then, for any non-zero complex b, for integer $N \geq 2$ and for f a transcendental meromorphic function, the equation

$$f^N(z)f'(z) = b$$

has infinitely many solutions in $\mathbf{C} - E$:

(i) if $\delta = \delta(\infty, f)$ and $\Delta = \Delta(\infty, f)$ satisfy $\delta(1 + 1/\Delta) > 1$;

or

(ii) if $\delta(1 + 1/\Delta) \leq 1$ and

$$(N+2)\delta^2 - (N + 4 + \frac{1}{2\Delta})\delta + 2 < 0; \tag{3.1}$$

or

(iii) if $\delta' = \delta(\infty, f') > 0$

or

(iv) if $\delta(\infty, f^N f') > 0$.

Although these alternative hypotheses appear somewhat cumbersome, the essential ingredient used in the proof is the condition (iv). We shall begin by showing that each of (i)-(iii) implies (iv) and then use this to establish the result.

The rather unwieldy quadratic condition in (ii) is satisfied when δ is greater than a value Λ which lies between $\frac{3}{2N+4}$ and $\frac{4}{2N+4}$.

If the deficiency is somewhat greater, we can obtain a result for discs. We prove:

Theorem 29 *Let $q > 1$ and let $\{a_m\}_{m=1}^{\infty}$ be a set of complex numbers such that*

$$\left|\frac{a_{m+1}}{a_m}\right| > q \tag{3.2}$$

for all m.

Let b be any non-zero complex number and $N \geq 2$ be an integer.
Let δ satisfy

$$\delta > \frac{4}{N+3}. \tag{3.3}$$

Then there exists ρ, $0 < \rho < \infty$, with the following property.

Suppose that f is a transcendental meromorphic function with

$$\delta(\infty, f) \geq \delta. \tag{3.4}$$

For each m, let d_m be given by

$$\log \frac{1}{d_m} = |a_m|^{\rho}, \tag{3.5}$$

and denote by B_m the open disc $\{z : |z - a_m| < d_m\}$, and let $E = \cup_{m=1}^{\infty} B_m$.

Then the equation

$$f^N(z) f'(z) = b$$

has infinitely many solutions in $\mathbf{C} - E$.

In the course of proving these theorems, we establish a number of results (Lemmas 42, 43 and 44) which concern the circumstances in which we can conclude that $\delta(\infty, f^N f') > 0$ and which may be of interest in their own right. In Chapter 4, to show that these lemmas are not redundant, we exhibit an example to show that there exist meromorphic functions f where

$$\delta(\infty, f) > 0, \quad \delta(\infty, f^N f') = 0. \tag{3.6}$$

Chapter 4 also considers the case when $N = 1$.

We begin by defining

$$F(z) = \frac{1}{N+1} f(z)^{N+1} \tag{3.7}$$

so that

$$F'(z) = f(z)^N f'(z). \tag{3.8}$$

3.2 Results required in the proofs

Theorem 30 (Valiron [29]) *Suppose that g is a meromorphic function and that $T(r,g) = O((\log r)^2)$. Then for any distinct $a, b \in \mathbf{C}^*$, we have*

$$T(r,g) = (1 + o(1)) \max\{N(r,a,g), N(r,b,g)\}.$$

Theorem 31 (Hayman [10]) *Suppose that g is a meromorphic function and that $T(r,g) = O((\log r)^2)$. Then*

$$T(r,g) \leq (2 + o(1)) T(r,g').$$

Theorem 32 (Toppila [28] - Theorem 7) *Let g be a transcendental meromorphic function such that*

$$\lim_{r\to\infty} \frac{T(2r,g)}{T(r,g)} = 1 \tag{3.9}$$

and $\delta = \delta(\infty, g) > 0$. Let $\Delta = \Delta(\infty, g)$.
If $\delta(1 + 1/\Delta) > 1$ then $\Delta(0, g') = 0$ and $\delta(\infty, g') > 0$.
If $\delta(1 + 1/\Delta) \le 1$ then

$$\Delta(0, g') \le \frac{1 - \delta(1 + 1/\Delta)}{1 - \delta} \limsup_{r\to\infty} \frac{N(r,g)}{N(r,g')}.$$

Note that if

$$T(r,g) = O((\log r)^2) \tag{3.10}$$

holds, then so does (3.9).

Since $1/\Delta \ge 1$, an immediate consequence of Toppila's theorem is:

Corollary 33 *Let g be a meromorphic function satisfying (3.9) and having $\delta(\infty, g) > 1/2$.*

Then $\delta(\infty, g') > 0$.

Toppila [27] gives an example to show that this is sharp ie g satisfying (3.9) with $\delta(\infty, g) = 1/2$ and $\delta(\infty, g') = 0$.

Theorem 34 (see eg Zalcman's survey paper [30]) *Let h be a meromorphic function and $N \geq 2$ an integer. Suppose that $h^N(z)h'(z)$ does not take the value 1 for $z \in B(a,1)$, for some complex number a.*
Then h belongs to a family which is normal on $B(a,1)$ and, in particular, the spherical derivative

$$h^{\#}(z) = \frac{|h'(z)|}{1+|h(z)|^2}$$

satisfies

$$h^{\#}(z) \leq K$$

on $B(a,1/2)$, where K is a constant independent of h and a.

Theorem 35 (see [8]: p11-13) *Let f be a meromorphic function. Let $T_0(r,f)$, the* Ahlfors-Shimizu characteristic *of f, be given by*

$$T_0(r,f) = \int_0^r \int_0^t \int_0^{2\pi} \frac{\rho}{\pi t} f^{\#}(\rho e^{i\theta})^2 d\theta d\rho dt. \tag{3.11}$$

Then

$$|T(r,f) - T_0(r,f) - \log^+|f(0)|| \leq \frac{1}{2}\log 2.$$

Theorem 36 (Anderson/Clunie [3]) *Let g be a meromorphic function with $\delta(\infty, g) > d > 0$ and suppose that $T(r,g) = O((\log r)^2)$.*
Then $\log|g(z)| > dT(r,g)$ for all z outside an ε-set (ie a set of discs $B(b_j, r_j)$ such that $b_j \to \infty$ and $\sum_{j=1}^{\infty} \frac{r_j}{|b_j|} < \infty$).

Theorem 37 (Milloux's inequality - see eg [8]: p57) *Let g be meromorphic and non-constant. Let ψ be a non-constant function of the form*

$$\psi(z) = \sum_{j=0}^{k} c_j g^{(j)}(z), \tag{3.12}$$

where the c_j are complex constants and $c_k \neq 0$, $k \geq 1$..

Then we have

$$T(r,g) < \overline{N}(r,g) + N(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{\psi - 1}) - N_0(r,\frac{1}{\psi'}) + S(r,g), \tag{3.13}$$

*where $N_0(r, 1/\psi')$ counts only those zeros of ψ' which are not **1**-points of ψ.*

Theorem 38 (Boutroux-Cartan lemma - see eg [11] p366) *Let $\{w_j\}_{j=1}^{n}$ be a set of n complex numbers. Let $d > 0$. Then*

$$\left| \prod_{j=1}^{n} (z - w_j) \right| \geq d^n \tag{3.14}$$

outside at most n discs in the z-plane of total diameter at most $4ed$.

The following is a simple well-known fact, which we set out as a lemma for convenience.

Lemma 39 *With F as in (3.7),*

$$\delta(\infty, f) = \delta(\infty, F), \quad \Delta(\infty, f) = \Delta(\infty, F). \tag{3.15}$$

To prove this, we note first that $n(r,F) = (N+1)n(r,f)$ for all r and hence $N(r,F) = (N+1)N(r,f)$ for all r.

Now, whether or not $|f(z)| \geq 1$, we have

$$(N+1)\log^+|f(z)| = \log^+|f(z)^{N+1}|.$$

So, using the simple fact (see eg [8, p5]) that

$$\log^+ ab \leq \log^+ a + \log^+ b,$$

we see that

$$\begin{aligned}
(N+1)\log^+|f| &= \log^+|f^{N+1}| \\
&\leq \log^+|f^{N+1}/(N+1)| + \log^+(N+1) \\
&\leq \log^+|f^{N+1}| + \log^+(1/(N+1)) + \log^+(N+1) \\
&= (N+1)\log^+|f| + \log^+(N+1),
\end{aligned}$$

ie $|(N+1)\log^+|f| - \log^+|F||$ is bounded by a constant.

So

$$m(r,F) = (N+1)m(r,f) + S(r,f), \quad T(r,F) = (N+1)T(r,f) + S(r,f),$$

and the result follows. Lemma 39 is proved.

Theorem 40 (Cauchy's integral formula/estimate for derivatives) *Let f be analytic in a domain D, and*

$$|f(z)| \leq M, \quad \{z : |z-\alpha| \leq r\} \subset D$$

for some constant M.

Then

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_{|z-\alpha|=r} \frac{f(\zeta)}{(\zeta-\alpha)^{n+1}}d\zeta.$$

Therefore also

$$|f^{(n)}(\alpha)| \leq \frac{n!M}{r^n}.$$

Theorem 41 (Rouché's Theorem) *Let f and g be functions meromorphic in a domain D. Let C be a simple closed curve in D and assume that the interior X of C lies in D. Suppose that*

$$|g(z)| < |f(z)| < \infty, \quad z \in C.$$

Then if $N(h)$ denotes the number of zeros of a meromorphic function h in X, and $P(h)$ denotes the number of poles of h in X, in both cases counting multiplicities, we have

$$N(f+g) - P(f+g) = N(f) - P(f).$$

3.3 Some preliminary results regarding $\delta(\infty, f^N f')$

We shall prove three lemmas concerning circumstances under which we can conclude that $\delta(\infty, F') > 0$.

Lemma 42 *If $\delta = \delta(\infty, f) > \frac{3}{N+3}$ then $\delta(\infty, F') > 0$.*

We have

$$\begin{aligned} T(r,F) &\leq T(r,F') + T(r,F/F') \\ &\leq T(r,F') + T(r,f/f') + O(1) \\ &= T(r,F') + T(r,f'/f) + O(1) \\ &\leq T(r,F') + \overline{N}(r,f) + \overline{N}(r,1/f) + S(r,f) \end{aligned}$$

and

$$\overline{N}(r,f) \leq \frac{1}{N+1} N(r,F) \leq \frac{1-\delta}{N+1} T(r,F) + S(r,F)$$
$$\overline{N}(r,1/f) \leq \frac{1}{N+1} N(r,1/F) \leq \frac{1}{N+1} T(r,F) + O(1)$$

so we get

$$T(r,F') \geq \frac{N-1+\delta}{N+1}T(r,F) + S(r,F). \tag{3.16}$$

Also

$$\begin{aligned} N(r,F') = N(r,F) + \overline{N}(r,F) \\ \leq \frac{N+2}{N+1}N(r,F) \\ \leq \frac{N+2}{N+1}(1-\delta)T(r,F) + S(r,F) \end{aligned}$$

and so

$$\limsup_{r\to\infty} \frac{N(r,F')}{T(r,F')} \leq \frac{(N+2)(1-\delta)}{N-1+\delta} < 1 \tag{3.17}$$

as long as $\delta > \frac{3}{N+3}$. Lemma 42 is proved.

Lemma 43 *Let* $\Delta = \Delta(\infty, f)$. *Suppose that* $T(r,f) = O((\log r)^2)$ *and that*

(i) $\delta(1+1/\Delta) > 1$; *or*

(ii) $\delta(1+1/\Delta) \leq 1$ *and*

$$(N+2)\delta^2 - (N+4+\frac{1}{2\Delta})\delta + 2 < 0. \tag{3.18}$$

Then $\delta(\infty, F') > 0$.

Recall first that Δ is as defined at (1.13) ie

$$\Delta = \Delta(\infty, f) = \limsup_{r\to\infty} \frac{m(r,f)}{T(r,f)}.$$

Note that the condition (3.18) in (ii) holds when

$$1 \geq \delta > \frac{N+4+\frac{1}{2\Delta} - \sqrt{N^2 + \frac{N+4}{\Delta} + \frac{1}{4\Delta^2}}}{2N+4} = \Lambda(N,\Delta) > 0,$$

since the other root of the quadratic on the left-hand side of (3.18) is easily seen to be greater than 1.

By observing that

$$(N+4+\frac{1}{2\Delta}-4)^2 = N^2+\frac{N}{\Delta}+\frac{1}{4\Delta^2} = (N^2+\frac{N+4}{\Delta}+\frac{1}{4\Delta^2})-\frac{4}{\Delta}$$

we see that

$$(N+4+\frac{1}{2\Delta})-4 < \sqrt{N^2+\frac{N+4}{\Delta}+\frac{1}{4\Delta^2}}$$

and so that

$$\Lambda(N,\Delta) < \frac{4}{2N+4}.$$

On the other hand, note that the first condition of (ii) requires

$$\Delta \geq \frac{\delta}{1-\delta}.$$

The left-hand side of (3.18) is, for $\delta > 0$, an increasing function of Δ and so the set of $\delta > 0$ satisfying (3.18) is contained in the corresponding set with the choice

$$\Delta = \frac{\delta}{1-\delta}.$$

Therefore, putting $\Delta = \frac{\delta}{1-\delta}$ in (3.18), we see that $\Lambda \geq x_0$, where x_0 is the zero between 0 and 1 of the quadratic

$$(N+2)x^2-(N+4+\frac{1-x}{2x})x+2 = (N+2)x^2-(N+\frac{7}{2})x+\frac{3}{2}.$$

But then $x_0 = \frac{3}{2N+4}$ and so we conclude that

$$\frac{3}{2N+4} \leq \Lambda < \frac{4}{2N+4}.$$

For the proof of the lemma, note that in either (i) or (ii) we have $\delta > 0$. Then, using Theorem 30, we have $T(r,f) = (1+o(1))N(r,1/f)$ and so, using Theorem 31, we have

$$\begin{aligned} T(r,F') &\geq N(r,1/F') + O(1) \\ &= NN(r,1/f) + N(r,1/f') + O(1) \\ &\geq NT(r,f) + (1-\Delta(0,f'))T(r,f') - S(r,f) \\ &\geq (N + \frac{1}{2} - \frac{1}{2}\Delta(0,f') - o(1))T(r,f). \end{aligned}$$

We also have

$$N(r,F') \leq (N+2)N(r,f) \leq (N+2)(1-\delta)T(r,f) + S(r,f)$$

and so

$$\frac{N(r,F')}{T(r,F')} \leq \frac{(N+2)(1-\delta)}{N+\frac{1}{2}-\frac{1}{2}\Delta(0,f')}(1+o(1)). \tag{3.19}$$

We now use Theorem 32.

In case (i), the result is immediate by taking $g = F$ (since $\delta(\infty,f) = \delta(\infty,F)$ and $\Delta(\infty,f) = \Delta(\infty,F)$ from Lemma 39).

In case (ii), taking $g = f$, we have

$$\Delta(0,f') \leq \frac{1-\delta(1+1/\Delta)}{1-\delta} \tag{3.20}$$

using the fact that $\frac{N(r,f)}{N(r,f')} \leq 1$. So then, combining (3.19) and (3.20), F' has deficient poles as long as δ satisfies

$$(N+2)\delta^2 - (N+4+\frac{1}{2\Delta})\delta + 2 < 0.$$

Lemma 43 is proved.

Lemma 44 *Let g be a meromorphic function of finite order, such that*

$$\limsup_{r\to\infty} \frac{T(r,g)}{T(r,g')} < \infty. \tag{3.21}$$

For a positive integer N set $G = \frac{1}{N+1} g^{N+1}$ so that $G' = g^N g'$.
Then

$$\delta(\infty, g') \leq \delta(\infty, G') \leq \delta(\infty, g). \tag{3.22}$$

Since g is of finite order, we have that

$$m(r, g'/g) = S(r,g) = o(T(r,g))$$

as $r \to \infty$ through all values of r, with no exceptional set (recalling (1.17) and the comment preceding it).

We shall use a number of times the elementary inequality

$$\frac{a}{a+b} \leq \frac{a+c}{a+b+c}$$

for $a, b, c \geq 0$.

We have

$$\begin{aligned}
\frac{m(r,G')}{T(r,G')} &\leq \frac{(N+1)m(r,g)+m(r,g'/g)}{(N+1)m(r,g)+m(r,g'/g)+(N+1)N(r,g)+\overline{N}(r,g)} \\
&= \frac{m(r,g)+S(r,g)}{T(r,g)+\frac{1}{N+1}\overline{N}(r,g)+S(r,g)} \\
&\leq (1+o(1))\frac{m(r,g)}{T(r,g)}
\end{aligned}$$

and so

$$\delta(\infty,G') \leq \delta(\infty,g).$$

Using (3.21), we have that

$$\frac{S(r,g)}{T(r,g')} = \frac{S(r,g)}{T(r,g)}\frac{T(r,g)}{T(r,g')} \to 0$$

as $r \to \infty$.

Also,

$$(N+1)m(r,g') \leq m(r,G') + Nm(r,g'/g)$$

and so

$$m(r,G') \geq (N+1)m(r,g') - S(r,g).$$

So then

$$\begin{aligned}
\frac{N(r,G')}{T(r,G')} &\leq \frac{(N+1)N(r,g')}{(N+1)N(r,g')+(N+1)m(r,g')-S(r,g)} \\
&= \frac{N(r,g')}{T(r,g')+S(r,g')}
\end{aligned}$$

and so we have

$$\delta(\infty,g') \leq \delta(\infty,G').$$

Lemma 44 is proved.

Corollary 45 *Let g be as in Lemma 44. If $\delta(\infty, g') > 0$, then $\delta(\infty, g) > 0$ and $\delta(\infty, G') > 0$.*

Remark: Note that if $T(r,g) = O((\log r)^2)$, then Theorem 31 shows that g satisfies the conditions of Lemma 44 and therefore the result of that lemma and the corollary hold for any such g.

3.4 Proof of Theorem 28

Recall that we set $F = \frac{1}{N+1} f^{N+1}$ so that $F' = f^N f'$. By considering f/b_0, where $b_0^{N+1} = b$, we may suppose without loss of generality that $b = 1$.

We suppose (for a contradiction) that all the 1-points of F' are at the points a_m.

By Milloux's inequality (Theorem 37) with $\psi = F'$, we have

$$T(r,F) < \overline{N}(r,F) + N(r,1/F) + \overline{N}(r, \frac{1}{F'-1}) - N_0(r, 1/F'') + S(r,F) \quad (3.23)$$

where $N_0(r, 1/F'')$ counts only those zeros of F'' which are not 1-points of F'.

Now

$$\overline{N}(r,F) = \overline{N}(r,f) \leq N(r,f) = \frac{1}{N+1} N(r,F) \leq \frac{1-\delta}{N+1} T(r,F) + S(r,F)$$

since $\delta(\infty, f) = \delta(\infty, F)$ (Lemma 39). Similarly

$$\overline{N}(r, 1/F) \leq \frac{1}{N+1} N(r, 1/F) \leq \frac{1}{N+1} T(r,F) + O(1).$$

Also

$$N(r,1/F) - N_0(r,1/F'') = 2\overline{N}(r,1/F) - N_1(r,1/F'')$$

where $N_1(r, 1/F'')$ counts only the zeros of F'' which are not zeros of F or 1-points of F', since F has no simple zeros.

Putting all this together, (3.23) becomes

$$\frac{N-2+\delta}{N+1}T(r,F) < \overline{N}\left(r, \frac{1}{F'-1}\right) - J(r,F'') + S(r,F) \tag{3.24}$$

where $J(r, F'')$ is a non-negative term relating to the zeros of F''.

Next, we note that, since all the 1-points of F' are at the a_m, and since $m = O(\log|a_m|)$, we have

$$\overline{n}(r, \frac{1}{F'-1}) = O(\log r)$$

and therefore

$$\overline{N}\left(r, \frac{1}{F'-1}\right) \leq \sum_{|a_m|<r} \log\frac{r}{|a_m|} \leq O((\log r)^2)$$

and so (3.24) gives $T(r, F) \leq O((\log r)^2)$ provided $N > 2$ or $\delta > 0$.
Note next, that

$$T(r, F'-1) = T(r, F') + S(r, F) \leq 2T(r, F) + S(r, F) \leq O((\log r)^2)$$

also in this case.

We shall now show that F' has deficient poles in each of the cases (i)-(iv).

Case (iv) is immediate. Cases (i) and (ii) follow from Lemma 43, the fact that $\delta > 0$ necessarily in either case, and the remark above that $\delta > 0$ implies that $T(r, f) \leq O((\log r)^2)$.

For $N > 2$, Case (iii) follows from the fact that $T(r, f) \leq O((\log r)^2)$, from the remark above, together with Lemma 44.

For Case (iii) and $N = 2$, we shall first have to show that $T(r, f) \leq O((\log r)^2)$ anyway, which will prove the assertion.

We choose a sequence $r_m \to \infty$ such that

$$U \geq |\frac{r_{m+1}}{r_m}| \geq V > 1, \quad m(r_m, f'/f) = o(T(r_m, f))$$

for all m, where U, V are constants, and such that the circles $|z| = r_m$ do not meet the discs $B(a_n, \varepsilon|a_n|)$, where $\varepsilon > 0$ is small, fixed.

We claim that

$$\hat{\delta} = \liminf_{m\to\infty} \frac{m(r_m, f)}{T(r_m, f)} > 0.$$

For, if not, then there is a subsequence of the r_m through which $m(r_m, f) = o(T(r_m, f))$.

For these m, using the fact that $\delta(\infty, f') > 0$,

$$\begin{aligned} T(r_m, f) &= (1 + o(1))N(r_m, f) \leq (1 + o(1))N(r_m, f') \leq (1 + o(1))T(r_m, f') \\ &\leq cm(r_m, f') \leq c[m(r_m, f) + m(r_m, f'/f)] = o(T(r_m, f)) \end{aligned}$$

(for some $c > 0$), which is a contradiction. So $\hat{\delta} > 0$ as claimed.

But now, at (3.24), we have

$$\frac{N - 2 + \hat{\delta}}{N + 1} T(r_m, F) < \overline{N}\left(r_m, \frac{1}{F' - 1}\right) - J(r_m, F'') + S(r_m, F)$$

and therefore $T(r_m, f) = O((\log r_m)^2)$.

But now, for any r, we have $r_m \le r < r_{m+1}$ for some m, and then

$$\begin{aligned} T(r,f) &\le T(r_{m+1},f) \le c_1(\log r_{m+1})^2 \le c_1(\log U r_m)^2 \\ &\le c_2(\log r_m)^2 \le c_2(\log r)^2 \end{aligned}$$

and so $T(r,f) = O((\log r)^2)$ as required, and now Lemma 44 gives us that F' has deficient poles.

In Case (iv), we can repeat the previous argument, using F' instead of f', to show that $T(r,f) = O((\log r)^2)$ necessarily in this case also.

So we have in all Cases (i)-(iv) that $\delta(\infty, F') > 0$ and $T(r,f) = O((\log r)^2)$, and therefore also that $T(r,F') = O((\log r)^2)$. Hence also, by Lemma 44, we have $\delta(\infty, f) > 0$ in all cases.

Now, by Theorem 36, we have that

$$\log|F'(z) - 1| > \frac{1}{2}\delta(\infty, F')T(|z|, F')$$

as $|z| \to \infty$ outside an ε-set round the a_m. In particular $|F'(z) - 1| > 10$, say, for $|z|$ sufficiently large outside such a set.

So, by Rouché's theorem, F' has the same number of zeros and 1-points, $k_m \ge 1$, say, inside each disc $B(a_m, \varepsilon|a_m|)$, for every $m \ge M$ for some sufficiently large M, and for some small fixed $\varepsilon > 0$.

Fix $m \ge M$. Since the only 1-point of F' inside $B(a_m, \varepsilon|a_m|)$ is at a_m, we have that the multiplicity of the zero of $F' - 1$ at a_m is k_m.

The number of loops γ, each beginning and ending at a_m, such that $|F'| = 1$ on each γ, is therefore k_m.

By considering instead (if necessary) loops γ on which $|F'(z) - \eta| = 1 - \eta$, for some small $\eta > 0$, we can ensure that the k_m loops γ do not intersect (except at a_m).

Together, the loops γ enclose a bounded open subset J of the plane. We define $D = J - \cup\gamma$, so that D is the union of a number of open components D_j, within each of which we have either $|F' - \eta| < 1 - \eta$ or $|F' - \eta| > 1 - \eta$.

Further, the number of D_j is k_m.

We claim that each D_j must contain either a pole or a zero of F'. For, suppose not. Then F' is analytic and non-zero inside D_j and $|F' - \eta| = 1 - \eta$ on the boundary of D_j, and so by the maximum and minimum modulus principles, $|F' - \eta| = 1 - \eta$ in the whole of D_j and so F' is constant, which is a contradiction.

Let p be the number of poles of f, q the number of zeros of f and s the number of zeros of f' in D, in each case counting multiplicity. Let $\overline{p}$, $\overline{q}$, $\overline{s}$ be the corresponding quantities ignoring multiplicity.
Then we have

$$Nq + s = k_m$$

since all zeros of F' in $B(a_m, \varepsilon|a_m|)$ are inside D. Also,

$$k_m \leq \overline{p} + \overline{q} + \overline{s}$$

using the fact that each D_j must contain either a zero or pole of F'.

So

$$Nq + s \leq \overline{p} + \overline{q} + \overline{s}$$

and so we obtain

$$(N-1)q \leq Nq - \overline{q} + s - \overline{s} \leq \overline{p} \leq p.$$

This holds for all large m and so, since all large zeros of f are contained within a $B(a_m, \varepsilon|a_m|)$, we have that

$$T(r,f) \leq (1+o(1))N(r,1/f) \leq (1+o(1))\frac{1}{N-1}N(r,f)+S(r,f) \leq (1+o(1))\frac{1-\delta}{N-1}T(r,f)$$

which is a contradiction, since $\delta > 0$. Theorem 28 is proved.

3.5 Proof of Theorem 29

Our method is a modification of that used in [14] and [15].

Suppose that f is transcendental and meromorphic in the plane and satisfies (3.4), where δ satisfies (3.3). By considering the function $\frac{f(z)}{b_0}$, where $b_0^{N+1} = b$, we may without loss of generality set $b = 1$. Recall that we defined

$$F(z) = \frac{1}{N+1} f^{N+1}(z) \tag{3.25}$$

so that

$$F'(z) = f^N(z) f'(z). \tag{3.26}$$

As in the proof of Theorem 28, our starting point is Milloux's inequality (Theorem 37). We have

$$T(r,F) < \overline{N}(r,F) + N(r,1/F) + N(r,\frac{1}{F'-1}) - N(r,1/F'') + S(r,F). \tag{3.27}$$

Now

$$\overline{N}(r,F) = \overline{N}(r,f) \leq N(r,f) = \frac{1}{N+1}N(r,F) \leq \frac{1-\delta}{N+1}T(r,F) + S(r,F) \tag{3.28}$$

since $\delta(\infty,f) = \delta(\infty,F)$. Similarly

$$\overline{N}(r,1/F) \leq \frac{1}{N+1}N(r,1/F) \leq \frac{1}{N+1}T(r,F) + O(1). \tag{3.29}$$

Also

$$N(r,1/F) - N(r,1/F'') = 2\overline{N}(r,1/F) - N_1(r,1/F''), \tag{3.30}$$

where $N_1(r,1/F'')$ counts only the zeros of F'' which are not zeros of F, since F has no zeros of multiplicity less than $N+1$.

Using (3.28),(3.29) and (3.30), (3.27) becomes

$$\frac{N-2+\delta}{N+1}T(r,F) < N(r,\frac{1}{F'-1}) - N_1(r,\frac{1}{F''}) + S(r,F). \tag{3.31}$$

We note that the restriction on δ in (3.3) implies that

$$\begin{aligned}(N+2)(1-\delta) &= N+2-(N+3)\delta+\delta \\ &< N+2-(N+3)\frac{4}{N+3}+\delta \\ &= N-2+\delta\end{aligned}$$

and so

$$\frac{(N+2)(1-\delta)}{N-2+\delta} < 1. \tag{3.32}$$

Therefore, we may choose constants k, Q with

$$1 < k < \frac{N+3}{N+2} < 3, \quad 1 < Q < q^{1/8}, \tag{3.33}$$

both sufficiently close to 1 that

$$(1+\frac{\log Q}{\log(q^{1/2}/Q)})k\frac{(N+2)(1-\delta)}{N-2+\delta} < 1. \tag{3.34}$$

3.5.1 f has finite order

This subsection is the work of Jim Langley. We show that if there exists a counter-example f to Theorem 29, then it must have finite order.

Proposition 46 *Let q, N, δ and $\{a_m\}$ be as in the statement of Theorem 29. Then there exists ρ, $0 < \rho < \infty$, such that if f is meromorphic in the plane and satisfies*

$$\delta(\infty, f) > 0, \tag{3.35}$$

and all but finitely many solutions of

$$f^N(z)f'(z) = 1$$

lie in

$$\cup_{m=1}^{\infty} B(a_m, 1),$$

then the order $\rho(f)$ of f satisfies

$$\rho(f) < \rho. \tag{3.36}$$

In this section, we use C_j to denote positive constants.

We use the notation $C(\alpha, R)$ to denote the circle $\{z : |z - \alpha| = R\}$.

Lemma 47 *There exists $C_1 > 0$ with the following property. Let g be meromorphic in $|z| \leq 1$, with $T(1, g) \leq T < \infty$ and $g(0) \neq \infty$. Then there exists $r^* \in [\frac{1}{2}, \frac{3}{4}]$ with*

$$\log |g(z)| \leq C_1 T$$

for $|z| = r^*$, and also

$$n(\frac{3}{4}, g) \leq C_1 T. \tag{3.37}$$

The last part is immediate, since we have

$$(\log \frac{4}{3}) n(\frac{3}{4}, g) \leq N(1, g) \leq T.$$

For the first part, let $w_1, \ldots, w_n$ be the poles of g in $|z| \leq \frac{3}{4}$, repeated according to multiplicity, with $n = n(\frac{3}{4}, g) \leq C_2 T$ from the part just proved. We set

$$h(z) = g(z) \prod_{j=1}^{n} \frac{z - w_j}{1 - \bar{w}_j z} \tag{3.38}$$

so that h is analytic in $|z| \leq \frac{3}{4}$.

For $|z| \leq \frac{3}{4}$, we have $|h(z)| \leq |g(z)|$, since

$$\begin{aligned} |1 - \bar{w}_j z|^2 - |z - w_j|^2 &= (1 - \bar{w}_j z)(1 - w_j \bar{z}) - (z - w_j)(\bar{z} - \bar{w}_j) \\ &= 1 + |w_j|^2 |z|^2 - |z|^2 - |w_j|^2 \\ &= (1 - |z|^2)(1 - |w_j|^2) \\ &> 0 \end{aligned}$$

and so

$$\left| \frac{z - w_j}{1 - \bar{w}_j z} \right| < 1.$$

Now, for $|z| \leq \frac{5}{8}$ we have, using the Poisson-Jensen formula,

$$\begin{aligned} \log |h(z)| &\leq C_3 T(\frac{3}{4}, h) \\ &= C_3 m(\frac{3}{4}, h) \\ &\leq C_3 m(\frac{3}{4}, g) \\ &\leq C_3 T(\frac{3}{4}, g) \leq C_3 T. \end{aligned} \tag{3.39}$$

We use the Boutroux-Cartan lemma (Theorem 38), with $d = (64e)^{-1}$. We get

$$|\prod_{j=1}^{n} (z - w_j)| \geq (\frac{1}{64e})^n \tag{3.40}$$

outside at most n discs of diameter at most $\frac{1}{16}$, and therefore we can find $r^* \in (\frac{1}{2}, \frac{5}{8})$ such that the circle $C(0, r^*)$ does not intersect any of these discs. So, on $|z| = r^*$ we have from (3.37), (3.38), (3.39) and (3.40) that

$$\begin{aligned} \log|g(z)| &\leq \log|h(z)| + n\log(64e) + \sum_{j=1}^{n} \log|1 - \bar{w}_j z| \\ &\leq C_3 T + n(\log 64e + \log 2) \\ &\leq C_4 T \end{aligned}$$

as required, and so Lemma 47 is proved.

Next, we suppose that g is a function meromorphic in $|z| < 2R$, for some $R \geq 1$, with $g^N(z)g'(z) \neq 1$ there, and $g(0) \neq \infty$. We set

$$h(z) = R^{-1/(N+1)} g(Rz) \tag{3.41}$$

so that

$$h^N(z)h'(z) = g^N(Rz)g'(Rz) \neq 1$$

in $B(0, 2)$.

Thus h belongs to a family of functions which is normal on $B(0, 2)$ (Theorem 34) and so

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2} \leq C_5$$

on $|z| \leq 1$, where C_5 is independent of h.

Thus the Ahlfors-Shimizu characteristic $T_0(r, h)$ of h satisfies (using (3.11))

$$T_0(r, h) \leq C_6, \quad 0 \leq r \leq 1,$$

and therefore, for such r, by Theorem 35,

$$T(r, h) \leq C_7 + \log^+ |g(0)|,$$

where C_7 is independent of g and h.

Now we can apply Lemma 47 to h. We obtain $r^* \in [\frac{1}{2}, \frac{3}{4}]$ such that

$$\log|h(z)| \leq C_8 + C_9 \log^+ |g(0)|, \quad |z| = r^*$$

and so, from the definition of h at (3.41),

$$\log|g(z)| \leq C_8 + C_9 \log^+ |g(0)| + C_{10} \log R, \quad |z| = r^* R. \tag{3.42}$$

Also, the last part of Lemma 47 gives us that

$$n(\frac{3}{4}R, g) = n(\frac{3}{4}, h) \leq C_{11} + C_{12} \log^+ |g(0)|. \tag{3.43}$$

Now let f be as in the statement of Proposition 46. Assume without loss of generality that a_1 is large.

Lemma 48 *There exists a path $\gamma' \to \infty$ with*

$$\log^+ \log^+ |f(z)| = O(\log|z|), \quad z \in \gamma'. \tag{3.44}$$

We construct a simple path $\gamma \to \infty$ which consists of

(i) the segments

$$\arg z = \arg a_n + \pi, \quad q^{1/2}|a_{n-1}| \leq |z| \leq q^{1/2}|a_n|$$

and

(ii) arcs of the circles $C(0, q^{1/2}|a_n|)$.

Then γ lies in $|z| \geq e$, since $|a_1|$ is assumed large. Also, γ does not meet any of the discs $B(a_m, 1)$ and so $f^N(z)f'(z) \neq 1$ on or near γ.

We fix a small positive σ, and choose $z_0 \in \gamma$ with $f(z_0) \neq \infty$. We choose points $z_n \in \gamma$ and radii $\hat{r}(z_n)$,

$$\sigma|z_n| \leq \hat{r}(z_n) \leq \frac{3}{2}\sigma|z_n|, \tag{3.45}$$

inductively as follows.

Assuming that $z_0, \ldots, z_n$ have been chosen, apply (3.42) with $g(z) = f(z-z_n)$, $R = 2\sigma|z_n|$ and $\hat{r} = r^* R$ to obtain that

$$\begin{aligned} \log|f(z)| &\leq C_8 + C_9 \log^+|f(z_n)| + C_{10} \log 2\sigma|z_n| \\ &\leq C_{13}(\log|z_n| + \log^+|f(z_n)|), \quad |z - z_n| = \hat{r}(z_n). \end{aligned} \tag{3.46}$$

Here we use the fact that $\log|z| \geq 1$ on γ.

Now we follow the path γ from z_n towards ∞, until the first point of intersection with the circle $C(z_n, \hat{r}(z_n))$, which we designate z_{n+1}.

Claim 1:

The circles $C(z_n, \hat{r}(z_n))$ and $C(z_{n+1}, \hat{r}(z_{n+1}))$ have non-empty intersection.

This is true since $z_{n+1} \in C(z_n, \hat{r}(z_n))$ and (3.45) gives

$$\begin{aligned} \hat{r}(z_{n+1}) &\leq \frac{3}{2}\sigma|z_{n+1}| \\ &\leq \frac{3}{2}\sigma(1 + \frac{3}{2}\sigma)|z_n| \\ &< 2\sigma|z_n| \leq 2\hat{r}(z_n), \end{aligned}$$

since σ was chosen to be small.

Claim 2:

We have $\lim_{n\to\infty} z_n = \infty$ *and*

$$n = O(\log|z_n|) \tag{3.47}$$

as $z_n \to \infty$.

Since $\hat{r}(z_n) \geq \sigma|z_n|$ and z_{n+1} lies on $C(z_n, \hat{r}(z_n))$, the number of z_n on any $C(0, q^{1/2}|a_m|)$ is uniformly bounded. So there exists a fixed j such that

$$|z_{n+j}| > (1+\sigma)|z_n|$$

for all n. So then

$$|z_n| > (1+\sigma)^{\frac{n-n_0}{j}}|z_{n_0}| \to \infty$$

as $n \to \infty$. Further,

$$n < n_0 + j(\log(1+\sigma))^{-1}\log\frac{|z_n|}{|z_{n_0}|} = O(\log|z_n|).$$

Claim 3:

We have

$$\log|f(z)| \leq (n+1)C_{13}^{n+1}\log|z_n| + C_{13}^{n+1}\log^+|f(z_0)|, \quad |z-z_n| = \hat{r}(z_n). \tag{3.48}$$

This follows by induction on n, using (3.46) and $|z_n| \geq |z_{n-1}|$.

From Claim 1 we see that the union $\hat{U}$ of the circles $C(z_n, \hat{r}(z_n))$ is connected, and from Claim 2, $z_n \to \infty$. We may therefore choose a simple path $\gamma' \to \infty$ lying in $\hat{U}$.

Claim 4:

We have

$$\log|f(z)| \leq |z|^{C_{14}}, \quad z \in \gamma'.$$

This follows at once from (3.47) and (3.48). This proves Lemma 48.

Lemma 49 *Let τ be a small positive constant. Then for*

$$q^{1/8}|a_m| \leq r \leq q^{-1/8}|a_{m+1}|$$

with m large, there exists a simple closed curve γ_r lying in

$$(1-\tau)r \leq |z| \leq (1+\tau)r,$$

with winding number 1 about 0, such that

$$\log|f(z)| \leq |z|^{C_{15}}, \quad z \in \gamma_r. \tag{3.49}$$

For the proof, we proceed as in Lemma 48, with modifications.
Assume that σ is positive, but small compared to τ. Choose $z_0 \in \gamma' \cap C(0, r)$. We then construct the sequence z_n, $\hat{r}(z_n)$ as in the proof of Lemma 48, so that (3.45) and (3.46) hold, but this time with z_{n+1} chosen so that

$$z_{n+1} \in C(0, r) \cap C(z_n, \hat{r}(z_n)).$$

It is evident that (3.48) continues to hold, and at most C_{16} iterations (where C_{16} depends only on σ) are required in order that $C(0,r)$ lie in the union of the closed discs $|z - z_n| \le \hat{r}(z_n)$.
Since Claim 1 continues to hold, and z_0 lies on γ', we obtain γ_r as a union of arcs of the circles $C(z_n, \hat{r}(z_n))$, and (3.49).

Lemma 50 *Let*

$$q^{1/4}|a_n| \le R < S \le q^{-1/4}|a_{n+1}|.$$

The number of poles of f in $R \le |z| \le S$ is at most S^C, for some fixed $C > 0$.

To prove this, we choose ξ with $R \le |\xi| \le S$ which lies on a γ_r (recall that γ_r are defined for all r with $q^{1/8}|a_m| \le r \le q^{-1/8}|a_{m+1}|$ for some m).
Then

$$\log|f(\xi)| \le |\xi|^{C_{15}}.$$

Using (3.43) with $g(z) = f(z - \xi)$, we obtain that the number of poles of f inside the disc, centre ξ, radius $\min(2\tau|\xi|, \frac{1}{2}(q^{1/4} - 1)|a_n|)$, is at most

$$C_{11} + C_{12}\log^+|f(\xi)| \le |\xi|^{C_{17}} \le S^{C_{17}}.$$

But since we can cover the annulus $R \le |z| \le S$ with at most $O(S^2)$ such discs, we obtain that the total number of poles of f in the annulus is no more than $S^{C_{18}}$, as required.
Lemma 50 is proved.

We are now in a position to show that f has finite order.
We take R with $q^{1/2}|a_n| \le R \le q^{-1/2}|a_{n+1}|$. Let

$$\Gamma_1 = \gamma_{q^{-1/8}R}, \quad \Gamma_2 = \gamma_{q^{1/8}R}.$$

Let

$$h(z) = f(z)\prod_{j=1}^{m}(z - w_j) \tag{3.50}$$

where the w_j are the poles of f between Γ_1 and Γ_2. So h is analytic between those curves.

On the Γ_j, using (3.49),

$$\log|f(z)| \leq R^{C_{19}}. \tag{3.51}$$

So, using (3.50), (3.51) and Lemma 50,

$$\log|h(z)| \leq R^{C_{19}} + mC_{20}\log R \leq R^{C_{21}}$$

on the Γ_j, and therefore between the Γ_j also, by the maximum modulus theorem.

By the Boutroux-Cartan lemma (Theorem 38) with $d = 1$, we have that

$$|\prod_{j=1}^{m}(z - w_j)| \geq 1$$

outside at most m discs of total diameter at most $4e$.

So we may choose $R' \in (R, R + 5e)$ such that $C(0, R')$ meets none of these discs and for $|z| = R'$ we have

$$\log|f(z)| \leq \log|h(z)| \leq R^{C_{21}} \leq R'^{C_{21}}.$$

Here we use the fact that $C(0, R')$ separates Γ_1 from Γ_2, since τ was chosen small.

But now, since $\delta(\infty, f) > 0$, we have

$$T(R', f) \leq \frac{2}{\delta}m(R', f) \leq \frac{2}{\delta}R'^{C_{21}} \leq R'^{C_{22}}.$$

But now

$$T(R,f) \leq T(R',f) \leq R'^{C_{22}} \leq R^{C_{23}}.$$

So the desired result holds for R satisfying $q^{1/2}|a_n| \leq |z| \leq q^{-1/2}|a_{n+1}|$. Suppose that r satisfies $q^{-1/2}|a_n| \leq |z| \leq q^{1/2}|a_n|$. Then

$$T(r,f) \leq T(q^{1/2}|a_n|,f) \leq (q^{1/2}|a_n|)^{C_{23}} \leq (q^{-1/2}|a_n|)^{C_{24}} \leq r^{C_{24}}.$$

This is the required result - equation (3.36) holds. Proposition 46 is proved.

3.5.2 Main part of the proof of Theorem 29

Assume now that q, δ, $\{a_n\}$ are as in the statement of the theorem, and that ρ is as in Proposition 46. Let the $\{d_n\}$ and $\{B_n\}$ be as given in the statement of the theorem, using this ρ. Suppose that f is transcendental and meromorphic in the plane, and satisfies (3.4), and define F as in (3.7).

Suppose that all but finitely many solutions of $F'(z) = 1$ are in the B_n. We shall aim for a contradiction, which will prove the theorem.

Denote by p_n the number of 1-points of F' in B_n, and by t_n the number of zeros of F'' in $|z - a_n| < 1$ which are not zeros of F (in each case counting multiplicity). Set

$$v_n = p_n - t_n. \tag{3.52}$$

Let y_n be the number of poles of F' (counting multiplicity) in the annulus

$$A_n = \{z; Q^{-1}|a_n| < |z| < Q|a_n|\}, \tag{3.53}$$

where Q is as chosen at (3.33).

Choose R_n with

$$q^{1/2}|a_n| \leq R_n < Q^{-1}|a_{n+1}|. \tag{3.54}$$

Since all the large 1-points of F' are in the B_n, we have from (3.31), for a fixed m_0 and large n, that

$$\frac{N-2+\delta}{N+1}T(R_n,F) < O(\log R_n) + \sum_{m=m_0}^{n} v_m \log\frac{R_n}{|a_m|} + o(\sum_{m=m_0}^{n}(p_m+t_m)). \tag{3.55}$$

Note that we may assume that $v_m \geq 0$ for all $m \geq m_0$, by deleting from the right hand side of the inequality those m for which this is not the case. We set

$$N(R_n) = \sum_{m=m_0}^{n} v_m \log\frac{R_n}{|a_m|}. \tag{3.56}$$

Now,

$$\begin{aligned}\sum_{m=m_0}^{n} p_m &\leq n(|a_n|+1, \frac{1}{F'-1}) \leq N(R_n, \frac{1}{F'-1})(\log R_n/(|a_n|+1))^{-1} \\ &\leq N(R_n, \frac{1}{F'-1})(\log\frac{q^{1/2}}{Q})^{-1} \leq (2+o(1))T(R_n,F)(\log\frac{q^{1/2}}{Q})^{-1}\end{aligned}$$

and similarly

$$\sum_{m=m_0}^{n} t_m \leq N(R_n, 1/F'')(\log\frac{q^{1/2}}{Q})^{-1} \leq (4+o(1))T(R_n,F)(\log\frac{q^{1/2}}{Q})^{-1}$$

for large enough n, and so

$$o(\sum_{m=m_0}^{n}(p_m+t_m)) = o(T(R_n,F)) = S(R_n,F).$$

Here we use the fact that f has finite order.

Thus, (3.55) becomes, using (3.56),

$$\frac{N-2+\delta}{N+1}T(R_n,F) \le (1+o(1))N(R_n). \tag{3.57}$$

Next, with k as in (3.33), let

$$J_n = \{m : m_0 \le m \le n; v_m > ky_m\}, \quad J_n' = \{m : m_0 \le m \le n; v_m \le ky_m\}. \tag{3.58}$$

Then we have (with obvious notation)

$$N(R_n) = N_{J_n}(R_n) + N_{J_n'}(R_n). \tag{3.59}$$

Now suppose that $m \in J_n'$. If w is a pole of F' in A_m, it will contribute

$$\log R_n/|w| = \log R_n/|a_m| + \log |a_m|/|w| \tag{3.60}$$

to $N(R_n, F')$. But (3.53) gives $Q^{-1} < \frac{|a_m|}{|w|} < Q$ and so $|\log \frac{|a_m|}{|w|}| < \log Q$ and we have, using (3.56), (3.58) and (3.60),

$$\begin{aligned} N_{J_n'}(R_n) &\le k \sum_{m=m_0}^{n} y_m \log \frac{R_n}{|a_m|} \\ &\le k[N(R_n, F') + n(Q|a_n|, F') \log Q] \\ &\le k(1 + \frac{\log Q}{\log \frac{q^{1/2}}{Q}})N(R_n, F') \\ &= \tilde{k} N(R_n, F') \end{aligned} \tag{3.61}$$

say.

Now, for any r,

$$\begin{aligned} N(r, F') = N(r, F) + \overline{N}(r, F) &\le N(r, F) + \frac{1}{N+1}N(r, F) \\ &\le \frac{(N+2)(1-\delta) + o(1)}{N+1}T(r, F) \end{aligned} \tag{3.62}$$

and therefore, from (3.57), (3.61) and (3.62), we have

$$
\begin{aligned}
N(R_n) - N_{J_n}(R_n) = N_{J_n'}(R_n) &\leq \tilde{k}N(R_n, F') \\
&\leq \tilde{k}\frac{(N+2)(1-\delta)+o(1)}{N+1}T(R_n, F) \\
&< \tilde{k}\frac{(N+2)(1-\delta)+o(1)}{N-2+\delta}N(R_n)
\end{aligned}
$$

(using the fact that $\log r = o(T(r, F))$ since F is transcendental), and so

$$
N(R_n) \leq [1 - \tilde{k}\frac{(N+2)(1-\delta)+o(1)}{N-2+\delta}]^{-1}N_{J_n}(R_n), \tag{3.63}
$$

where the coefficient on the right hand side is positive, for sufficiently large n, from (3.34).

Now, using (3.57) again, we have

$$
\begin{aligned}
T(R_n, F) &< \left(\frac{N+1}{N-2+\delta} + o(1)\right)N(R_n) \\
&\leq \frac{N+1}{N-2+\delta}[1 - \tilde{k}\frac{(N+2)(1-\delta)+o(1)}{N-2+\delta}]^{-1}N_{J_n}(R_n) \\
&\leq \lambda(\delta, N)N_{J_n}(R_n), \qquad (3.64)
\end{aligned}
$$

say, for n sufficiently large, where $\lambda = \lambda(\delta, N)$ is a positive constant.

Now, let

$$
J = \{m : m \geq m_0; v_m > ky_m\} = \cup_{n=m_0}^{\infty} J_n. \tag{3.65}
$$

The fact that F is transcendental, together with (3.64), implies that $v_m > 0$ for infinitely many $m \in J$. So then, whether or not the v_m are bounded above, there exists $m_1 \in J$ and infinitely many $M \in J$ such that

$$
v_M = \max\{v_m : m \in J, m_1 \leq m \leq M\}. \tag{3.66}
$$

For such M, and R_M defined as in (3.54), we have from (3.56) and (3.64),

$$T(R_M, F) < \lambda N_{J_M}(R_M) \leq \lambda M v_M \log \frac{R_M}{|a_1|} + O(\log R_M).$$

But

$$|a_M| > q|a_{M-1}| > \ldots > q^{M-1}|a_1|$$

and so

$$M < (1 + o(1))(\log q)^{-1} \log |a_M|$$

and so we have

$$T(R_M, F) < \lambda(\log q)^{-1} v_M (1 + o(1))(\log R_M)^2 \tag{3.67}$$

and

$$T(R_M, F' - 1) < 2\lambda(\log q)^{-1} v_M (1 + o(1))(\log R_M)^2. \tag{3.68}$$

In particular, this is true for $R_M = q^{1/2}|a_M|$, by (3.54), so that

$$T(q^{1/2}|a_M|, F' - 1) < \lambda' v_M (\log |a_M|)^2, \tag{3.69}$$

for some constant $\lambda' > 0$, when M satisfies (3.66).

Recall that, since $M \in J$, we have by (3.52) and (3.58),

$$p_M = v_M + t_M > k y_M. \tag{3.70}$$

The main part of the proof is contained in:

Lemma 51 *Fix a large positive constant K_1. Suppose that M is a large positive integer, that*

$$p_M > ky_M \tag{3.71}$$

and

$$T(q^{1/2}|a_M|, F' - 1) < K_1 p_M (\log |a_M|)^2. \tag{3.72}$$

Then there exists W_M with

$$\frac{-k}{4}|a_M|^\rho \leq \log W_M \leq \frac{-1}{4}|a_M|^\rho \tag{3.73}$$

such that

$$F'(z) = 1 + o(1), \quad |z - a_M| = W_M. \tag{3.74}$$

If, in addition, $M \in J$, then we have also

$$v_M = p_M - t_M < \frac{k}{k-1}. \tag{3.75}$$

Let K_1 and M be as given in the statement of the lemma. Let $z_1, \ldots, z_{p_M}$ be the 1-points of F' in B_M and $w_1, \ldots, w_{y_M}$ be the poles of F' in A_M. We set

$$P(z) = \prod_{j=1}^{p_M}(z - z_j), \quad \Pi(z) = \prod_{l=1}^{y_M}(z - w_l), \quad h(z) = [F'(z) - 1]P(z)^{-1}\Pi(z) \tag{3.76}$$

so that h is analytic and non-zero in A_M.

We apply the Poisson-Jensen formula (Theorem 8) to h in the disc $B(0, r_M)$ for $r_M = Q|a_M|$. We get

$$|\log|h(z)|| \leq \frac{r_M + |z|}{r_M - |z|}[m(r_M, h) + m(r_M, 1/h)] + \sum_\zeta \log\left|\frac{r_M^2 - \bar{\zeta}z}{r_M(\zeta - z)}\right| \tag{3.77}$$

where the sum is over all zeros and poles ζ of h in $|z| < r_M$, using the fact that

$$\left|\frac{r_M(\zeta - z)}{r_M^2 - \bar{\zeta}z}\right| \leq 1.$$

Since all the ζ lie in $B(0, Q^{-1}|a_M|)$, we have

$$|P(z)| > 1, \quad |\Pi(z)| > 1, \quad |z| = r_M,$$

and so $m(r_M, 1/P) = m(r_M, 1/\Pi) = 0$, and therefore

$$m(r_M, h)+m(r_M, 1/h) \leq m(r_M, F'-1)+m(r_M, 1/(F'-1))+m(r_M, P)+m(r_M, \Pi). \tag{3.78}$$

Now, using (3.71),

$$m(r_M, P) + m(r_M, \Pi) \leq (p_M + y_M)\log 2r_M < (1 + 1/k)p_M \log 2r_M. \tag{3.79}$$

Also, for z satisfying $|z - a_M| \leq 4$, since h has no zeros or poles in A_M, we have for every zero or pole ζ of h that

$$|\zeta - a_M| \geq (1 - Q^{-1})|a_M|$$

and so

$$\left|\frac{r_M^2 - \bar{\zeta}z}{r_M(\zeta - z)}\right| \leq \frac{2r_M^2}{r_M((1 - Q^{-1})|a_M| - 4)} \leq \frac{3Q}{1 - Q^{-1}} = c_1, \tag{3.80}$$

say, and so we deduce that from (3.77), (3.78), (3.79) and (3.80), for $|z-a_M| \leq 4$, that

$$|\log|h(z)|| \leq c_2[2T(r_M, F'-1)+(1+1/k)p_M \log 2r_M]+(\log c_1)\left(n(r_M, h) + n(r_M, 1/h)\right) \tag{3.81}$$

for a suitable constant c_2.

But, recalling from (3.33) that

$$1 < Q < q^{1/8},$$

we have

$$\begin{aligned} n(r_M, h) + n(r_M, 1/h) &\leq n(r_M, F' - 1) + n(r_M, 1/(F' - 1)) \\ &\leq (\log q^{3/8})^{-1}\left(N(q^{1/2}|a_M|, F' - 1) + N(q^{1/2}|a_M|, 1/(F' - 1))\right) \\ &\leq (2(\log q^{3/8})^{-1} + O(1))T(q^{1/2}|a_M|, F' - 1) \end{aligned} \tag{3.82}$$

and so, from (3.71), (3.72), (3.81) and (3.82), we deduce that

$$|\log|h(z)|| \leq c_3 p_M (\log|a_M|)^2, \quad |z - a_M| \leq 4 \tag{3.83}$$

for some positive constant $c_3 = c_3(\delta, N)$.

We note next that

$$\begin{aligned} y_M &\leq n(q^{1/4}|a_M|, F') \\ &\leq (N+2)n(q^{1/4}|a_M|, f) \\ &\leq (N+2)(\log q^{1/4})^{-1} N(q^{1/2}|a_M|, f) \\ &\leq (N+2)(\log q^{1/4})^{-1}(1-\delta/2)T(q^{1/2}|a_M|, f) \\ &= o(|a_M|^\rho) \\ &= o(\log \frac{1}{d_M}) \end{aligned} \tag{3.84}$$

by (3.5) and (3.36).

Let the interval I_M be given by

$$I_M = [\frac{-k}{4}|a_M|^\rho, \frac{-1}{4}|a_M|^\rho], \tag{3.85}$$

in which k is as in (3.33), and consider the set

$$S_M = \{z : \log|z - a_M| \in I_M\}.$$

We note that

$$B_M \cap S_M = \emptyset,$$

since

$$\log d_M = -|a_M|^\rho < \frac{-k}{4}|a_M|^\rho$$

by (3.5), using $k < 3$ from (3.33).

Also, $S_M \subset A_M$.

So S_M contains at most y_M poles of F', and so there exists a sub-interval $I'_M = [\alpha, \beta] \subset I_M$ of length at least

$$L_M = \frac{(k-1)}{4} \frac{|a_M|^\rho}{y_M + 1} \tag{3.86}$$

such that

$$S'_M = \{z : \log|z - a_M| \in I'_M\} \tag{3.87}$$

contains no poles of F'.

We define U_M, V_M by

$$\log U_M = \alpha, \quad \log V_M = \beta \tag{3.88}$$

so that F' has no poles in

$$U_M \leq |z - a_M| \leq V_M.$$

Set

$$W_M = \frac{U_M + V_M}{2}. \tag{3.89}$$

We observe that, by (3.5), (3.33), (3.85) and (3.88),

$$\log d_M = -|a_M|^\rho \leq \frac{4}{3}\alpha = \frac{4}{3}\log U_M = \log U_M^{4/3}$$

and so

$$d_M = o(U_M), \tag{3.90}$$

since $U_M \to 0$ as $M \to \infty$.

Also, using (3.5), (3.84), (3.86) and (3.88),

$$\log \frac{V_M}{U_M} = \beta - \alpha \geq L_M \to \infty$$

as $M \to \infty$, and so

$$U_M = o(V_M). \tag{3.91}$$

So

$$W_M = (\frac{1}{2} + o(1))V_M. \tag{3.92}$$

Thus, by (3.85), (3.88) and (3.92),

$$\frac{1}{W_M} = (2 - o(1))\frac{1}{V_M} = (2 - o(1))\exp(-\log V_M) \geq \exp(\frac{1}{4}|a_M|^\rho) \geq |a_M|^\rho. \tag{3.93}$$

Note in particular that

$$\log W_M \leq \frac{-1}{4}|a_M|^\rho \to -\infty \tag{3.94}$$

as $M \to \infty$.

Now, by (3.76),

$$\left|\frac{P'}{P}(z)\right| = \left|\sum_{j=1}^{p_M} \frac{1}{z - z_j}\right| \geq \frac{1}{|z - a_M|} Re(\sum_{j=1}^{p_M} \frac{1}{1 - \frac{z_j - a_M}{z - a_M}}). \tag{3.95}$$

For $|z - a_M| = W_M$, we have, using (3.89) and (3.90),

$$\frac{|z_j - a_M|}{|z - a_M|} \leq \frac{d_M}{W_M} = o(1). \tag{3.96}$$

Thus, using the fact that

$$Re(\frac{1}{1-u}) \geq 1 - \frac{|u|}{1 - |u|}$$

for $|u| < 1$, we have from (3.95) and (3.96),

$$\left|\frac{P'}{P}(z)\right| \geq \frac{p_M}{W_M}(1 - o(1)), \quad |z - a_M| = W_M. \tag{3.97}$$

Also we have, for $|z-a_M| = W_M$, since the annulus S'_M defined at (3.87) contains no poles of F',

$$\left|\frac{\Pi'}{\Pi}(z)\right| = \left|\sum_{l=1}^{y_M} \frac{1}{z-w_l}\right| \leq \sum_{l=1}^{y_M} \frac{1}{|z-w_l|} \leq \frac{2y_M}{V_M - U_M} = \frac{(1+o(1))y_M}{W_M}, \quad (3.98)$$

by (3.89), (3.91) and (3.92), and so we obtain, using (3.71), (3.97) and (3.98),

$$\left|\frac{P'\Pi}{P\Pi'}(z)\right| \geq (1-o(1))\frac{p_M}{y_M} > k - o(1) \quad (3.99)$$

and so

$$\left|\frac{\Pi'}{\Pi}(z)\right| < (\frac{1}{k} + o(1))\left|\frac{P'}{P}(z)\right| \quad (3.100)$$

when $|z-a_M| = W_M$.

For such z, we also have, from (3.83), (3.93) and (3.97), and using Cauchy's estimate for derivatives (Theorem 40), that

$$\left|\frac{h'}{h}(z)\right| \leq c_4 p_M (\log|a_M|)^2 = o(\frac{p_M}{W_M}) = o(\left|\frac{P'(z)}{P(z)}\right|). \quad (3.101)$$

So, from (3.33), (3.100) and (3.101) we have, for $|z-a_M| = W_M$,

$$\left|\frac{\Pi'}{\Pi}(z)\right| + \left|\frac{h'}{h}(z)\right| < \left|\frac{P'}{P}(z)\right|. \quad (3.102)$$

Now, by (3.76),

$$F''(z)\Pi(z) = P(z)h(z)\left(\frac{P'(z)}{P(z)} + \frac{h'(z)}{h(z)} - \frac{\Pi'(z)}{\Pi(z)}\right) \quad (3.103)$$

and so, by Rouché's Theorem (Theorem 41), the number of zeros minus the number of poles of $F''\Pi$ inside the circle $|z-a_M| = W_M$ equals the number of zeros minus the number of poles of P' there, using (3.102) and the fact that h

is analytic and non-zero inside the annulus A_M.

The zeros of P' lie in the convex hull of the set of zeros of P (see eg [1, p29]), and so in B_M. Thus, P' has $p_M - 1$ zeros and no poles inside $C(a_M, W_M)$. Next, Π has no poles, and by (3.76) all its zeros are poles of $F'-1$ and therefore poles of $F''\Pi$. Hence F'' has at least $p_M - 1$ zeros in $|z - a_M| < W_M$.
We recall that t_M is the number of zeros of F'' in $B(a_m, 1)$ which are not zeros of F. But every zero of F is a zero of F' and F'', since $N + 1 \geq 3$, and we deduce that

$$t_M + s_M \geq p_M - 1, \tag{3.104}$$

where s_M denotes the number of zeros of F' in $|z - a_M| < W_M$.

However, for $|z - a_M| = W_M$, using (3.71), (3.76), (3.83), (3.89), (3.90), (3.92) and (3.94),

$$\begin{aligned} \log|F'(z) - 1| &= \log|P(z)| + \log|h(z)| - \log|\Pi(z)| \\ &\leq p_M \log(d_M + W_M) + c_3 p_M (\log|a_M|)^2 - y_M \log \frac{V_M - U_M}{2} \\ &= p_M(\log W_M + o(1)) - y_M(\log W_M + o(1)) + c_3 p_M (\log|a_M|)^2 \\ &\leq p_M((1 - 1/k)\log W_M + o(1) + c_3(\log|a_M|)^2) \\ &< 0. \end{aligned} \tag{3.105}$$

In fact, (3.94) implies that

$$\log|F'(z) - 1| \to -\infty, \quad |z - a_M| = W_M \tag{3.106}$$

as $M \to \infty$, so (3.74) holds.

By (3.105), the value of $w = F'(z)$ stays in the disc $|w - 1| < 1$ as z traverses the circle $C(a_M, W_M)$. So we can define a continuous branch of $\log w$ on a domain containing the image of that circle in the w-plane. So we have

$$\int_{C(a_M,W_M)} \frac{F''(z)}{F'(z)} = [\log F'(z)] = 0$$

and so, by the argument principle (Theorem 12), F' has the same number of zeros as poles in $|z - a_M| < W_M$. Therefore

$$s_M \leq y_M. \tag{3.107}$$

Now suppose in addition that $M \in J$. Then, by (3.58) and (3.65),

$$v_M > k y_M. \tag{3.108}$$

But now, from (3.52), (3.104), (3.107) and (3.108), we have

$$v_M = p_M - t_M \leq s_M + 1 \leq y_M + 1 < v_M/k + 1 \tag{3.109}$$

and so

$$v_M < \frac{k}{k-1}, \tag{3.110}$$

which is (3.75). Lemma 51 is proved.

Lemma 52 *We have*

$$T(r, F) = O((\log r)^2), \quad T(r, F' - 1) = O((\log r)^2) \tag{3.111}$$

as $r \to \infty$.

Suppose that M is large and satisfies (3.66). Then M satisfies the hypotheses of Lemma 51. For, (3.71) holds from (3.70), and (3.72) holds from (3.69) and

(3.70).

Therefore (3.73), (3.74) and (3.75) hold for M, and in particular

$$v_M < \frac{k}{k-1}.$$

But now, from the definition of M at (3.66), we conclude that

$$v_m < \frac{k}{k-1}$$

for all large $m \in J$.

Therefore, from (3.56), (3.59) and (3.64) we have

$$\begin{aligned} T(R_n, F) &< \lambda \frac{k}{k-1} \sum_{m \in J_n} \log \frac{R_n}{|a_m|} + O(\log R_n) \\ &\leq (n\lambda \frac{k}{k-1} + O(1)) \log R_n \end{aligned} \tag{3.112}$$

for all sufficiently large n and R_n satisfying $q^{1/2}|a_n| \leq < R_n < Q^{-1}|a_{n+1}|$.

But

$$|a_n| > q|a_{n-1}| > \ldots > q^{n-m_0}|a_{m_0}|$$

and so

$$n < m_0 + (\log q)^{-1} \log \frac{|a_n|}{|a_{m_0}|}$$

and therefore

$$T(R_n, F) \leq c_5 (\log R_n)^2 \tag{3.113}$$

for some constant $c_5 = c_5(\delta, N)$, for all sufficiently large n and for R_n within the given range.

But for r sufficiently large but outside this range, there exists n such that

$$Q^{-1}|a_n| \leq r \leq q^{1/2}|a_n|$$

and then

$$\begin{aligned} T(r,F) &\leq T(q^{3/4}|a_n|,F) \leq c_5(\log(q^{3/4}|a_n|))^2 \\ &\leq c_6(\log|a_n|)^2 \leq c_7(\log(Q^{-1}|a_n|))^2 \leq c_7(\log r)^2 \end{aligned} \tag{3.114}$$

for suitable positive constants c_6, c_7. Also,

$$T(r,F'-1) = T(r,F') + O(1) \leq 2T(r,F) + O(1) = O((\log r)^2),$$

and so Lemma 52 is proved.

From (3.3) and Lemma 42, we have $\delta(\infty, F') > 0$.
Therefore, since $T(r,F'-1) = O((\log r)^2)$, we may apply the Anderson/Clunie theorem (Theorem 36) to $F'-1$. We obtain that

$$|F'(z) - 1| > 10 \tag{3.115}$$

say, for $|z|$ sufficiently large and z outside $\cup_{n=1}^{\infty} B(a_n, \varepsilon|a_n|)$, for a small fixed $\varepsilon > 0$.
So, by Rouché's Theorem, F' has the same number of 1-points as zeros (counting multiplicities) inside each $B(a_n, \varepsilon|a_n|)$, for large enough n.
For large enough n, we have $B(a_n, d_n) \subset B(a_n, \varepsilon|a_n|)$.

We fix μ with

$$\frac{N+3}{N} < \mu < \frac{1}{1-\delta}, \tag{3.116}$$

which is possible by (3.3).

Let ξ_n denote the number of zeros of f in $B(a_n, \varepsilon|a_n|)$ and τ_n denote the number of poles of f in A_n, counting multiplicities in each case. Here A_n is as defined in (3.53). Recalling that $F' = f^N f'$ and that y_n denotes the number of poles of F' in A_n, we note that

$$y_n \leq (N+2)\tau_n. \tag{3.117}$$

We claim that there exist infinitely many n such that

$$\xi_n > \mu\tau_n. \tag{3.118}$$

For, if not, then since a zero of f is a zero of F', we have that all large zeros of f lie inside the $B(a_n, \varepsilon|a_n|)$ and then, using Valiron's result (Theorem 30), for $r = q^{1/2}|a_m|$, m large, we get

$$\begin{aligned}
T(r,f) &= (1+o(1))N(r,1/f) \\
&= (1+o(1))\sum \xi_n(\log\frac{r}{|a_n|} + o(1)) \\
&= (1+o(1))\sum \xi_n \log\frac{r}{|a_n|} + O(\log r) \\
&\leq \mu(1+o(1))\sum \tau_n \log\frac{r}{|a_n|} + O(\log r) \\
&\leq \mu(1+o(1))(N(r,f) + O(\log r)) \\
&\leq \mu(1-\delta+o(1))T(r,f) \\
&< T(r,f),
\end{aligned}$$

by (3.116), which is a contradiction.

We take a large n which satisfies (3.118). Then, from the remark following (3.115), the number of zeros of F' in $B(a_n, \varepsilon|a_n|)$ is p_n and we have

$$p_n \geq N\xi_n > N\mu\tau_n \geq \mu\frac{N}{N+2}y_n > \frac{N+3}{N+2}y_n > ky_n$$

by (3.33), (3.116) and (3.117), and so n satisfies (3.71). By Lemma 52, (3.72) is satisfied and so we can apply Lemma 51 to conclude, using (3.90) and (3.94), that

$$F'(z) = 1 + o(1), \quad |z - a_n| = W_n, \quad d_n = o(W_n), \quad W_n = o(1). \tag{3.119}$$

Consider the set

$$X = \{z \in B(a_n, \varepsilon|a_n|) : |F'(z)| < 1\}. \tag{3.120}$$

The boundary ∂X consists of finitely many Jordan curves $\hat{\gamma}_j$, not necessarily disjoint, but each containing a 1-point of F' and so meeting B_n. Every zero of F' in $B(a_n, \varepsilon|a_n|)$ lies inside at least one $\hat{\gamma}_j$.

We delete any $\hat{\gamma}_j$ for which there exists a j' with

$$\hat{\gamma}_j \subseteq \hat{\gamma}_{j'} \cup int(\hat{\gamma}_{j'}).$$

We are left with finitely many $\hat{\gamma}_j$, which we relabel γ_j.

Set

$$L = B(a_n, W_n) \cup \cup_j int(\gamma_j).$$

L is connected, since every γ_j meets B_n, which is contained in $B(a_n, W_n)$.

The external boundary of L consists of finitely many arcs Γ_j of level curves $|F'| = 1$, each lying in

$$W_n \leq |z - a_n| < \varepsilon|a_n|,$$

and finitely many arcs of $C(a_n, W_n)$ (we adjust W_n slightly, if necessary, to ensure that no γ_j meets $C(a_n, W_n)$ infinitely often).

We obtain finitely many domains D_j, each bounded by a Γ_j and an arc of $C(a_n, W_n)$, such that every zero of F' in

$$W_n < |z - a_n| < \varepsilon |a_n|$$

lies in one of the D_j.

As we traverse that part of the boundary of D_j which is Γ_j, we do not pass through any 1-point of F' (since all 1-points of F' lie in B_n), and so $\arg F'(z)$ cannot change by more than 2π.
As we pass along the remaining portion of the boundary of D_j (where $|z-a_n| = W_n$), we have $\log|F'(z) - 1| < 0$ by (3.74) and so $\arg F'(z)$ cannot change by more than π.
So as we traverse the whole boundary of D_j once, $\arg F'(z)$ cannot change by more than 3π, ie it must change by $\pm 2\pi$ or 0.
So, by the argument principle,

$$Z_j \leq \Sigma_j + 1 \tag{3.121}$$

where Z_j denotes the number of zeros of F' in D_j, and Σ_j the number of poles of F' in D_j.

Now let z_j denote the number of zeros and σ_j the number of poles of f in D_j respectively.
Then we have, using (3.26) and (3.121),

$$z_j \leq \frac{1}{N} Z_j \leq \frac{1}{N}(\Sigma_j + 1) \leq \frac{1}{N}((N+2)\sigma_j + 1) \leq \frac{N+3}{N}\sigma_j. \tag{3.122}$$

[The last inequality appears to require $\sigma_j > 0$. But if $\sigma_j = 0$ then we have $z_j \leq 1/N$ and hence $z_j = 0$, and so (3.122) still holds.]

So

$$\sum_j z_j \le \frac{N+3}{N} \sum_j \sigma_j. \tag{3.123}$$

Furthermore, we know from the remark preceding (3.107) that F' has the same number of zeros as poles in $|z - a_n| < W_n$, and so the zeros Z and poles Σ of f in that region are related by

$$Z \le \frac{N+2}{N} \Sigma. \tag{3.124}$$

Therefore, combining (3.123) and (3.124), the number of zeros of f in $B(a_n, \varepsilon |a_n|)$ is no greater than $\frac{N+3}{N}$ times the number of poles of f in that disc, and we have

$$\xi_n \le \frac{N+3}{N} \tau_n. \tag{3.125}$$

But now, using (3.116), (3.118) and (3.125), we have

$$\xi_n > \mu \tau_n \ge \mu \frac{N}{N+3} \xi_n > \xi_n,$$

which is a contradiction. Theorem 29 is proved.

Chapter 4

Further results for the problem $f^N f' = 1$

4.1 Introduction

In this chapter we continue our investigations of the problem $f^N f' = 1$ begun in Chapter 3. We shall present two results, as follows.

First, we shall consider the problem of finding analogues of Picard sets for $ff' = 1$, ie when $N = 1$. In contrast to the results of Chapter 3, we shall obtain a result for entire functions only (Theorem 54).

Second, we shall refer back to the lemmas of Section 3.3 regarding the circumstances in which the deficiency of f or f' at the poles enables us to conclude that $\delta(\infty, F') > 0$. In order to show that these results are not redundant, we give an example (Section 4.3) to show that it is possible to have a function f which is transcendental meromorphic, with

$$\delta(\infty, f) > 0, \quad \delta(\infty, f^N f') = 0, \tag{4.1}$$

for N any positive integer.

4.2 $N = 1$, f entire

Anderson, Baker and Clunie have proved [2] the following:

Theorem 53 *Let $E = \{a_m\}_{m=1}^{\infty}$ be a set of complex numbers such that*

$$\liminf_{m\to\infty} \frac{|a_{m+1}|}{|a_m| \log |a_m|} > c > 0 \tag{4.2}$$

for a constant c.

If f is a transcendental entire function, then $f'(z)f(z)$ takes every finite complex value, except possibly zero, infinitely often in $\mathbf{C} - E$.

In this section we improve the spacing condition on the a_m. We shall prove:

Theorem 54 *Suppose that $\varepsilon > 0$. Let $E = \{a_m\}_{m=1}^{\infty}$ be a set of complex numbers such that $a_m \to \infty$ and*

$$|a_m - a_{m'}| > \varepsilon |a_m|, \quad m \neq m'. \tag{4.3}$$

If f is a transcendental entire function, then $f'(z)f(z)$ takes every finite complex value, except possibly zero, infinitely often in $\mathbf{C} - E$.

4.2.1 Results required in the proof

Theorem 55 (see eg [21] section 3.55) *Let Γ be a simple closed level curve of an entire function f. Then Γ encloses one more zero of f than of f'.*

Theorem 56 (Barry [5]) *Let g be a transcendental entire function. Let*

$$m(r) = \min\{|g(z)| : |z| = r\}, \quad M(r) = M(r,g) = \max\{|g(z)| : |z| = r\}. \tag{4.4}$$

Suppose that $M(r)$ satisfies

$$\limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^2} \le \sigma, \tag{4.5}$$

for some constant $\sigma > 0$.

Then the set of r for which

$$\frac{m(r)}{M(r)} > \exp(-(2-\delta)\delta^{-1}\pi^2\sigma), \tag{4.6}$$

has lower logarithmic density at least $1-\delta$.

Lemma 57 (Anderson/Baker/Clunie [2]) *Let f be an entire function. Then*

$$T(r,f) = O\left(\overline{N}\left(r, \frac{1}{ff'-1}\right)\right) \tag{4.7}$$

as $r \to \infty$ outside a set of r of finite linear measure.

4.2.2 Proof of Theorem 54

Our proof is a refinement of the method of [2], with the key new ingredient coming from [5].

Suppose that the set E is as stated in the theorem, satisfying the condition (4.3) for some $\varepsilon > 0$. We may arrange that the a_m are numbered in order of non-decreasing modulus.

Let f be a transcendental entire function. We set

$$F(z) = \frac{1}{2} f(z)^2 \tag{4.8}$$

so that

$$F'(z) = f'(z) f(z). \tag{4.9}$$

We suppose that all but finitely many of the zeros of $F'(z) - 1$ lie in E and will aim for a contradiction, which will prove the theorem. We may without loss of generality suppose that $F'(a_m) = 1$ for all m, since we may discard any a_m for which this is not the case.

Throughout the proof, C_j will denote positive constants.

First, we establish an upper bound for the number of a_m in $B(0, r)$.

Lemma 58 *The number $n(r)$ of a_m with $|a_m| < r$ satisfies*

$$n(r) \leq C_1 \varepsilon^{-2} \log r. \tag{4.10}$$

Note first that the discs $B(a_m, \frac{1}{3}\varepsilon|a_m|)$ do not intersect. Each such disc has area

$$\frac{\pi}{9}\varepsilon^2|a_m|^2.$$

For any r, the annulus $r/2 \le |z| < r$ has area

$$\frac{3}{4}\pi r^2$$

and so can contain at most

$$C_2\varepsilon^{-2}$$

of the a_m, using the fact that $|a_m| \ge r/2$ for any such.

By considering the annuli $r/2 \le |z| < r$, $r/4 \le |z| < r/2, \ldots$, we may conclude that the number of a_m in $B(0, r)$ is at most

$$C_3 + \alpha C_2\varepsilon^{-2}, \tag{4.11}$$

where C_3 is the number of a_m in $B(0, 1)$ and α is an integer with $1/2 < r/2^\alpha \le 1$.

We have

$$\alpha < 1 + \frac{\log r}{\log 2} \tag{4.12}$$

Combining with (4.11) gives the required result. Lemma 58 is proved.

Lemma 59 *We have*

$$T(r, f) = O((\log r)^2) \tag{4.13}$$

outside an exceptional set of finite Lebesgue measure.

We establish an estimate for $\overline{N}\left(r, \frac{1}{F'-1}\right)$.

Suppose that

$$|a_n| < r \leq |a_{n+1}|.$$

Then

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{F'-1}\right) &= \sum_{j=1}^{n} \log \frac{r}{|a_j|} \\
&= \log \frac{r^n}{|a_n| \dots |a_1|} \\
&\leq n \log r \\
&\leq C_1 \varepsilon^{-2} (\log r)^2,
\end{aligned}$$

using Lemma 58.

Now the result follows from Lemma 57. Lemma 59 is proved.

Lemma 60 *The slow growth of f implies that the exceptional set in Lemma 59 does not occur, ie (4.13) holds for all large r.*

Suppose not, ie

$$T(r, f) \leq C(\log r)^2 \tag{4.14}$$

for some constant C, outside an unbounded set S of r of finite Lebesgue measure.

Take a large $r \in S$. We can find $\theta \in (1, r)$ such that $\theta r \notin S$. Then

$$T(r, f) \leq T(\theta r, f) \leq C(\log \theta r)^2 \leq 4C(\log r)^2$$

and so (4.14) holds for all r after all. Lemma 60 is proved.

Using (1.10), we conclude from Lemma 60 that there exists a constant $\sigma > 0$ such that

$$\log M(r, f) \leq \sigma(\log r)^2. \tag{4.15}$$

Since $T(r, f) = O((\log r)^2)$, and hence also $T(r, f') = O((\log r)^2)$ and $T(r, F') = O((\log r)^2)$, we may apply the Anderson/Clunie theorem (Theorem 36).

Lemma 61 *Let ε_1 be a positive constant, small compared to ε. For sufficiently large r, the disc $B(a_m, \varepsilon_1|a_m|)$ contains one more zero of f than of f'.*

We use the argument of Section 6 of [2].
By Theorem 36, there exists an epsilon-set, centred on the a_m, outside which F' is large. By increasing the radii of this epsilon-set, we may ensure that f, f' are both large also outside the set.

Let Δ be a disc of the epsilon-set which contains at least one zero of f and which is sufficiently remote that $|f(z)^2| > 6|z|$ on the boundary $\partial\Delta$. This occurs for all sufficiently remote Δ, using the Anderson/Clunie theorem and the fact that f is transcendental.

Suppose that Δ contains k zeros of f. Then it contains $2k$ zeros of f^2, and so $2k$ zeros of $f(z)^2 - 2z$, by Rouché's theorem.

The radius ρ of Δ is small, certainly $\rho < |a_m|/8$. So for any λ which satisfies

$$\frac{9}{4}|a_m| < \lambda < \frac{21}{8}|a_m|,$$

and any $z \in \Delta$, we have

$$2|z| < \lambda < 3|z|.$$

For such λ, the components of

$$A = \{z : |f(z)^2 - 2z| \leq \lambda\}$$

which meet $\bar{\Delta}$ lie in the interior of Δ, since for $z \in A$, $|f(z)^2| \leq 2|z| + \lambda < 5|z|$. We choose a λ satisfying the conditions above, but also such that the level set

$$\partial A = \{z : |f(z)^2 - 2z| = \lambda\}$$

does not contain a_m. Note that the choice of λ ensures that all zeros of f in Δ lie in A.

Since a_m is the only point in Δ at which

$$\frac{1}{2}\frac{d}{dz}(f(z)^2 - 2z) = f(z)f'(z) - 1 = 0,$$

∂A consists of finitely many disjoint simple closed loops. On each loop, $2|z| < \lambda = |f(z)^2 - 2z|$ and so by Rouché's theorem, the loop contains the same number of zeros of f^2 as of $f^2 - 2z$. Furthermore, this number of zeros must be positive.

By Theorem 55, each such loop contains one fewer zero of $ff' - 1$ than of $f^2 - 2z$ (and therefore of f^2).

But a_m is the only point in Δ at which $ff' - 1$ takes the value 0. Since each loop must contain an even number of zeros of f^2, and hence at least 1 zero of $ff' - 1$, there can only be one loop. This loop contains all $2k$ zeros of f^2 in Δ since, from the remark above, all such zeros lie in A.

So $ff' - 1$ has $2k - 1$ zeros in A, and hence in Δ, since all such zeros are at a_m.

By Rouché's theorem, ff' also has $2k - 1$ zeros in Δ. But now since f has k zeros in Δ, f' must have $k - 1$ zeros in Δ. So Δ contains one more zero of f

than of f'.

Since all large zeros of f, f' lie in the epsilon-set, and since the sets $B(a_m, \varepsilon_1|a_m|)$ contain the discs of any epsilon-set, for sufficiently large m, Lemma 61 is proved.

Lemma 62 *There exist $\varepsilon_1 > 0$ and $C_4 > 0$ such that the set S of r which satisfy (using the notation of Theorem 56)*

$$\frac{m(r)}{M(r)} > C_4 \tag{4.16}$$

and which are such that $|z| = r$ does not meet any $B(a_m, \varepsilon_1|a_m|)$, has lower logarithmic density at least $1/4$.

Given any $\varepsilon_1 > 0$, the logarithmic density of the set of r such that the circle $|z| = r$ meets at least one of the $B(a_m, \varepsilon_1|a_m|)$ is at most

$$\frac{n(r)}{\log r} \log \frac{1+\varepsilon_1}{1-\varepsilon_1} \leq C_1 \varepsilon^{-2} \log \frac{1+\varepsilon_1}{1-\varepsilon_1}, \tag{4.17}$$

using Lemma 58.

By choosing ε_1 sufficiently small, we can ensure that this density is at most $1/2$, say.

We apply Theorem 56 with $g = f$ and $\delta = 1/4$, σ as in (4.15), and we set

$$C_4 = \exp(-(2-\delta)\delta^{-1}\pi^2\sigma).$$

We conclude, using (4.17), that the set S of r which satisfy (4.16) and are such that $|z| = r$ does not meet any $B(a_m, \varepsilon_1|a_m|)$ has lower logarithmic density at least 1/4. Lemma 62 is proved.

Lemma 63 *For any positive A we have, for $\beta > 0$ and all sufficiently large r,*

$$M(r+\beta, f) \geq M(r, f)(1 + \frac{\beta}{r})^A. \tag{4.18}$$

Since f is transcendental, we have

$$\frac{\log M(r, f)}{\log r} \to \infty$$

as $r \to \infty$, from (1.10) and (1.15).

Further, we have that $\log M(r, f)$ is a convex function of $\log r$ (ie the curve of the graph of $\log M(r, f)$ against $\log r$ between any two points lies below a straight line joining those two points of the graph) - see eg [21] section 5.32.

We conclude that

$$\frac{d \log M(r, f)}{d \log r} \to \infty$$

as $r \to \infty$.

Given positive A we deduce that, for all sufficiently large r and $\beta > 0$,

$$\frac{\log M(r+\beta, f) - \log M(r, f)}{\log(r+\beta) - \log r} > A.$$

Rearranging, we obtain (4.18). Lemma 63 is proved.

We choose $\varepsilon_2 > 0$ which is small compared to ε_1.
We choose A so large that

$$(1 + \frac{\varepsilon_2}{1-\varepsilon_2})^A C_4 > 1. \tag{4.19}$$

Recall from Lemma 62 that the set S of r which satisfy (4.16) and which are such that $|z| = r$ does not meet any $B(a_m, \varepsilon_1 |a_m|)$ has lower logarithmic density at least 1/4.

We choose large $r, R \in S$ which satisfy, for at least one m,

$$r < |a_m| < R \tag{4.20}$$

and which also satisfy

$$m(r) \leq M(r) < \frac{1}{2} m(R) \leq \frac{1}{2} M(R). \tag{4.21}$$

This is possible since $M(\rho) \to \infty$ as $\rho \to \infty$, together with the fact from (4.16) that

$$\frac{m(r)}{M(r)} > C_4, \quad \frac{m(R)}{M(R)} > C_4. \tag{4.22}$$

Lemma 64 *Given ε_2, A, r, R as chosen above at (4.19), (4.20) and (4.21), we have*

$$n(R, 1/f) - n(r, 1/f) = n(R, 1/f') - n(r, 1/f'). \tag{4.23}$$

We consider a level curve Γ_r of $|f(z)| = M(r(1 - \varepsilon_2), f)$ which passes through a point z_0 on $|z| = r(1 - \varepsilon_2)$. Since ε_2 was chosen small compared to ε_1, there are no zeros of f or f' in $r(1 - \varepsilon_2) \leq |z| < r$ (recalling that f, f' are large outside an epsilon-set contained within the union of the $B(a_m, \varepsilon_1 |a_m|)$).

By the maximum modulus principle, Γ_r cannot enter the disc $|z| < r(1-\varepsilon_2)$. Also, we have, from Lemma 63, (4.19) and (4.22),

$$\begin{aligned} m(r) &> C_4 M(r) \\ &\geq C_4 M(r(1-\varepsilon_2))(1+\frac{\varepsilon_2}{1-\varepsilon_2})^A \\ &> M(r(1-\varepsilon_2)) \end{aligned}$$

and so we conclude that Γ_r cannot pass into $|z| \geq r$ either.

So Γ_r lies wholly in the annulus $r(1-\varepsilon_2) \leq |z| < r$. Further (by changing ε_2 very slightly, if necessary) we can ensure that Γ_r is a simple closed curve.

So the number of zeros of f inside Γ_r is

$$n(r, 1/f) = n((1-\varepsilon_2)r, 1/f).$$

Applying the same argument to f', we conclude that the number of zeros of f' inside Γ_r is $n(r, 1/f')$.

By Theorem 55, we conclude that there is one more zero of f than of f' inside Γ_r, and so

$$n(r, 1/f) - n(r, 1/f') = 1. \tag{4.24}$$

We may repeat the above argument for R instead of r to obtain Γ_R which encloses one more zero of f than of f', and so

$$n(R, 1/f) - n(R, 1/f') = 1. \tag{4.25}$$

Rearranging (4.24) and (4.25) gives (4.23). Lemma 64 is proved.

So the annulus $r < |z| < R$ contains equal numbers of zeros of f and f'.

But this region contains at least one a_m, from (4.20), and since all large zeros of f and f' are contained in the $B(a_m, \varepsilon_1 |a_m|)$ and each such disc contains one more zero of f than of f', by Lemma 61, we have obtained a contradiction. Theorem 54 is proved.

4.3 An example of transcendental meromorphic f with $\delta(\infty, f) > 0$, $\delta(\infty, f^N f') = 0$

In order to show that the results of Section 3.3 are not redundant, we exhibit an example to show that it is possible to have a function f which is transcendental meromorphic, with

$$\delta(\infty, f) > 0, \quad \delta(\infty, f^N f') = 0, \tag{4.26}$$

for N a positive integer.

We choose sequences r_k, s_k, a_k, b_k, n_k, and constants L, ε, β, γ as follows:

$$r_1 = 100, \quad a_0 = b_0 = 1 \tag{4.27}$$

$$L = 36N^2, \quad \gamma = \beta = 4N, \quad \varepsilon = (18N^2)^{-1} \tag{4.28}$$

$$r_k \geq r_{k-1}^L \tag{4.29}$$

$$n_k = L^k \tag{4.30}$$

$$a_k = a_{k-1} r_k^{n_{k-1} - n_k} \tag{4.31}$$

$$s_k = r_{k+1}^{\varepsilon} \tag{4.32}$$

$$b_k = b_{k-1} s_k^{n_{k-1} - n_k}. \tag{4.33}$$

We set

$$h(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \tag{4.34}$$

$$g(z) = \sum_{k=1}^{\infty} b_k z^{n_k}. \tag{4.35}$$

Note that, for $|z| = r$,

$$\left| \frac{a_k z^{n_k}}{a_{k-1} z^{n_{k-1}}} \right| = \left(\frac{r}{r_k} \right)^{n_k - n_{k-1}}, \tag{4.36}$$

using (4.31).

In particular, for $r_k \geq 2|z|$, we get

$$|a_k z^{n_k}| \leq \frac{1}{2} |a_{k-1} z^{n_{k-1}}| \tag{4.37}$$

and we deduce that both sums (4.34) and (4.35) converge.

We estimate $h(z)$ for $r_k \leq r = |z| < r_{k+1}$. Then $r_{k+2} > 2|z|$ and so, using (4.31) and (4.37),

$$\begin{aligned} \left| \sum_{p>k} a_p z^{n_p} \right| &\leq 2|a_{k+1} z^{n_{k+1}}| \\ &= 2a_{k+1} r^{n_{k+1}} \\ &= 2a_k r_{k+1}^{n_k - n_{k+1}} r^{n_{k+1}} \\ &= 2a_k r^{n_k} \left(\frac{r}{r_{k+1}} \right)^{n_{k+1} - n_k}. \end{aligned} \tag{4.38}$$

Similarly, $r_{k-1} < |z|/2$ and so (4.36) gives

$$|a_{p-1} z^{n_{p-1}}| < \frac{1}{2} |a_p z^{n_p}| \tag{4.39}$$

for $p \leq k - 1$, so that

$$\begin{aligned} \left| \sum_{p<k} a_p z^{n_p} \right| &\leq 2|a_{k-1} z^{n_{k-1}}| \\ &= 2a_{k-1} r^{n_{k-1}} \\ &= 2a_k r^{n_k} \left(\frac{r_k}{r} \right)^{n_k - n_{k-1}}. \end{aligned} \tag{4.40}$$

We obtain, therefore, that for $r_k \le r < r_{k+1}$ and $|z| = r$,

$$h(z) = a_k r^{n_k}(1 + \xi_k(z)), \tag{4.41}$$

where

$$|\xi_k(z)| \le 2((\frac{r_k}{r})^{n_k - n_{k-1}} + (\frac{r}{r_{k+1}})^{n_{k+1}-n_k}) < 4. \tag{4.42}$$

Similarly, for $s_k \le r < s_{k+1}$ and $|z| = r$,

$$g(z) = b_k r^{n_k}(1 + \zeta_k(z)), \tag{4.43}$$

where

$$|\zeta_k(z)| \le 2((\frac{s_k}{r})^{n_k - n_{k-1}} + (\frac{r}{s_{k+1}})^{n_{k+1}-n_k}) < 4. \tag{4.44}$$

Note that since $\varepsilon L = 2$, we have

$$r_k < r_k^2 \le s_k < r_{k+1} < r_{k+1}^2 \le s_{k+1} < \dots. \tag{4.45}$$

Lemma 65 *We have, for large k,*

$$\log M(r, h) \le n_k \log r, \quad r_k \le r < r_{k+1}, \tag{4.46}$$

$$\log M(r, g) \le n_k \log r, \quad s_k \le r < s_{k+1}. \tag{4.47}$$

We prove only the first inequality. By (4.41) and (4.42), for $r_k \le r < r_{k+1}$, k large,

$$|h(z)| \le 5a_k r^{n_k} \le r^{n_k}. \tag{4.48}$$

The result follows. Lemma 65 is proved.

Lemma 66 *We have, as $r \to \infty$,*

$$T(r,h) + T(r,g) = O((\log r)^2) \tag{4.49}$$

and

$$T(r,h) \sim \log M(r,h), \quad T(r,g) \sim \log M(r,g). \tag{4.50}$$

Suppose $r_k \leq r < r_{k+1}$. From (4.29),

$$\log r \geq \log r_k \geq L \log r_{k-1} \geq \ldots \geq L^k \frac{\log r_1}{L} \tag{4.51}$$

and so, from (4.30),

$$n_k = L^k = O(\log r_k) = O(\log r). \tag{4.52}$$

So, using (4.46),

$$T(r,h) \leq \log M(r,h) \leq n_k \log r = O((\log r)^2). \tag{4.53}$$

(4.49) is proved. But now (4.50) follows from (1.10).

Lemma 66 is proved.

Lemma 67 *We have, for large k,*

$$\frac{L-1}{2L} n_k \log s_k \leq \log \frac{a_k}{b_k} \leq \frac{L}{L-1} n_k \log s_k. \tag{4.54}$$

For the right hand inequality, we have, from (4.31) and (4.33),

$$\begin{aligned} \log \frac{a_k}{b_k} &= \log \frac{a_{k-1}}{b_{k-1}} + (n_k - n_{k-1}) \log \frac{s_k}{r_k} \\ &= \ldots \\ &= O(1) + \sum_{j=1}^{k} (n_j - n_{j-1}) \log \frac{s_j}{r_j} \\ &\leq \sum_{j=1}^{k} n_j \log s_j \\ &\leq \frac{L}{L-1} n_k \log s_k. \end{aligned} \tag{4.55}$$

For the left hand inequality, we first note that

$$r_k^2 = r_k^{\varepsilon L} \leq s_k$$

and so

$$r_k \leq s_k^{1/2}$$

and therefore

$$\frac{s_k}{r_k} \geq s_k^{1/2}. \tag{4.56}$$

So then, using (4.30), (4.31) and (4.33),

$$\begin{aligned} \frac{a_k}{b_k} &= \frac{a_{k-1}}{b_{k-1}} \left(\frac{s_k}{r_k} \right)^{n_k - n_{k-1}} \\ &\geq \frac{a_{k-1}}{b_{k-1}} s_k^{n_k(1-1/L)/2}. \end{aligned} \tag{4.57}$$

This shows that $a_k/b_k \to \infty$ as $k \to \infty$, and gives (4.54).

Lemma 67 is proved.

Now we set

$$f(z) = \frac{h(z)}{g(z)}. \tag{4.58}$$

Lemma 68 *We have*

$$\delta(\infty, f) > 0. \tag{4.59}$$

From (4.34) we have

$$h^{(n_k)}(0) = n_k! a_k. \tag{4.60}$$

Also, Cauchy's integral formula (Theorem 40) gives

$$h^{(n_k)}(0) = \frac{n_k!}{2\pi i} \int_{|z|=r} \frac{h(z)}{z^{n_k+1}} dz \tag{4.61}$$

and so we have

$$\begin{aligned} a_k r^{n_k} &= \frac{r^{n_k}}{2\pi}\left|\int_{|z|=r}\frac{h(z)}{z^{n_k+1}}dz\right| \\ &\leq \frac{r^{n_k}}{2\pi}2\pi r\frac{M(r,h)}{r^{n_k+1}} \\ &= M(r,h). \end{aligned} \tag{4.62}$$

Now, for $s_k \leq r < r_{k+1}$ we have, by (4.43), (4.44) and (4.62),

$$\begin{aligned} M(r,g) &\leq 5b_k r^{n_k} \\ &= 5\frac{b_k}{a_k}a_k r^{n_k} \\ &\leq 5\frac{b_k}{a_k}M(r,h). \end{aligned} \tag{4.63}$$

Therefore

$$\log M(r,g) + \log\frac{a_k}{b_k} \leq \log 5 + \log M(r,h). \tag{4.64}$$

Thus (4.32), (4.46), (4.50) and (4.54) give

$$\begin{aligned} m(r,f) &\geq m(r,h) - m(r,g) \\ &\geq T(r,h) - \log M(r,g) \\ &\geq (1-o(1))\log M(r,h) - \log M(r,g) \\ &\geq (1-o(1))\log\frac{a_k}{b_k} - o(T(r,g)) \\ &\geq (1-o(1))\frac{L-1}{2L}n_k\log s_k - o(T(r,g)) \\ &\geq (1-o(1))\frac{L-1}{2L}\varepsilon n_k\log r - o(T(r,g)) \\ &\geq \left(\frac{L-1}{2L}\varepsilon - o(1)\right)n_k\log r. \end{aligned} \tag{4.65}$$

Since

$$T(r,f) = O(n_k\log r), \tag{4.66}$$

by (4.46), we deduce

$$\liminf\frac{m(r,f)}{T(r,f)} > 0 \tag{4.67}$$

as $r \to \infty$ through $s_k \le r < r_{k+1}$.

Suppose now that $r_k \le r < s_k$. Then

$$M(r, g) \le 5b_{k-1}r^{n_{k-1}} \le 5\frac{b_{k-1}}{a_k}r^{n_{k-1}-n_k}M(r, h) \tag{4.68}$$

using the same analysis as in (4.63). So

$$\log M(r, g) \le \log 5 + \log \frac{b_{k-1}}{a_k} - (n_k - n_{k-1}) \log r + \log M(r, h)$$

and this time we have

$$\begin{aligned} m(r, f) &\ge (1 - o(1))((n_k - n_{k-1}) \log r + \log \frac{a_k}{b_{k-1}}) - o(T(r, g)) \\ &\ge (1 - o(1))(1 - \frac{1}{L})n_k \log r, \end{aligned} \tag{4.69}$$

and (4.67) follows, as $r \to \infty$ through $r_k \le r < s_k$.

Lemma 68 is proved.

Now suppose that z is such that

$$r_{k+1}^{2/(9N)} = s_k^{\beta} < r = |z| < r_{k+1}^{1/\gamma} = r_{k+1}^{1/(4N)}. \tag{4.70}$$

From (4.41), (4.43) and (4.58) we have

$$f(z) = \frac{a_k}{b_k}\frac{1 + \xi_k(z)}{1 + \zeta_k(z)} = \frac{a_k}{b_k}(1 + \frac{\xi_k(z) - \zeta_k(z)}{1 + \zeta_k(z)}) \tag{4.71}$$

where $\xi_k(z)$ and $\zeta_k(z)$ are analytic.

So

$$f'(z) = \frac{a_k}{b_k}\frac{(\xi_k'(z) - \zeta_k'(z))(1 + \zeta_k(z)) + (\zeta_k(z) - \xi_k(z))\zeta_k'(z)}{(1 + \zeta_k(z))^2}. \tag{4.72}$$

For z as above, from (4.42),

$$\begin{aligned} |\xi_k(z)| &\leq 2((\frac{r_k}{r})^{n_k-n_{k-1}} + (\frac{r}{r_{k+1}})^{n_{k+1}-n_k}) \\ &\leq 2((\frac{r_k}{s_k^\beta})^{n_k-n_{k-1}} + (\frac{r_{k+1}^{1/\gamma}}{r_{k+1}})^{n_{k+1}-n_k}) \\ &\leq 2(s_k^{-(\beta-1)(n_k-n_{k-1})} + r_{k+1}^{-(1-1/\gamma)(n_{k+1}-n_k)}). \end{aligned} \tag{4.73}$$

Now, using $s_k = r_{k+1}^\varepsilon$, $(n_{k+1} - n_k) = L(n_k - n_{k-1})$ and $\varepsilon(\beta - 1) < L(1 - 1/\gamma)$, we conclude that the first term dominates and therefore that

$$|\xi_k(z)| < 4s_k^{-(\beta-1)(n_k-n_{k-1})}. \tag{4.74}$$

Similarly, for the same z, by (4.44),

$$\begin{aligned} |\zeta_k(z)| &\leq 2((\frac{s_k}{r})^{n_k-n_{k-1}} + (\frac{r}{s_{k+1}})^{n_{k+1}-n_k}) \\ &\leq 2((\frac{s_k}{s_k^\beta})^{n_k-n_{k-1}} + (\frac{r_{k+1}^{1/\gamma}}{s_{k+1}})^{n_{k+1}-n_k}) \\ &\leq 2(s_k^{-(\beta-1)(n_k-n_{k-1})} + r_{k+1}^{-(1-1/\gamma)(n_{k+1}-n_k)}) \\ &< 4s_k^{-(\beta-1)(n_k-n_{k-1})}. \end{aligned} \tag{4.75}$$

Using the fact that ξ_k and ζ_k are analytic, and the Cauchy inequality, we conclude from (4.74) and (4.75) that

$$|\xi_k'(z)| < 4s_k^{-(\beta-1)(n_k-n_{k-1})} \tag{4.76}$$

$$|\zeta_k'(z)| < 4s_k^{-(\beta-1)(n_k-n_{k-1})} \tag{4.77}$$

for $s_k^\beta + 1 < |z| < r_{k+1}^{1/\gamma} - 1$.

Recall from (4.54) that for large k,

$$\log \frac{a_k}{b_k} \leq \frac{L}{L-1} n_k \log s_k. \tag{4.78}$$

For $s_k^\beta + 1 < |z| < r_{k+1}^{1/\gamma} - 1$, we have, from (4.71), (4.74), (4.75) and (4.78),

$$\log|f(z)| \leq \frac{L}{L-1} n_k \log s_k + O(1). \tag{4.79}$$

Also, from (4.72), (4.74), (4.75), (4.76), (4.77) and (4.78),

$$\begin{aligned} \log|f'(z)| &\leq \frac{L}{L-1} n_k \log s_k + \log 32 + \log 4 s_k^{-(\beta-1)(n_k - n_{k-1})} \\ &\leq O(1) + (\frac{L}{L-1} - (\beta - 1)(1 - \frac{1}{L})) n_k \log s_k. \end{aligned} \tag{4.80}$$

So, from (4.79) and (4.80),

$$\begin{aligned} \log|f^N(z) f'(z)| &= N \log|f(z)| + \log|f'(z)| \\ &\leq N \frac{L}{L-1} n_k \log s_k + O(1) + (\frac{L}{L-1} - (\beta-1)(1 - \frac{1}{L})) n_k \log s_k \\ &= O(1) + ((N+1)\frac{L}{L-1} - (\beta-1)(1-\frac{1}{L})) n_k \log s_k \qquad (4.81) \\ &< 0 \qquad (4.82) \end{aligned}$$

since $\beta = 4N$.

So, for infinitely many circles $r = |z|$, we have $\log|f^N(z) f'(z)| < 0$ and so

$$\frac{m(r, f^N f')}{T(r, f^N f')} = 0. \tag{4.83}$$

We conclude that

$$\delta(\infty, f^N f') = 0. \tag{4.84}$$

The function f has the required property.

Remark:

In the above example, we chose the constants L, ε, β, γ, α with a view to making the calculations relatively straightforward to follow, and to prove simply that an example exists with $\delta(\infty, f) > 0$, $\delta(\infty, f^N f') = 0$.

By choosing carefully alternative values for these constants, an example of the above form can be constructed with $\delta(\infty, f) = c$, for any $c \in (0, \frac{1}{N+3})$.

This compares interestingly with our result (Lemma 42) that $\delta(\infty, f) > \frac{3}{N+3}$ implies that $\delta(\infty, f^N f') > 0$.

Chapter 5

Analogues of Picard sets for the problem $f = 0,\ f^{(k)} = 1$

5.1 Introduction

In this chapter we shall consider analogues of Picard sets for the following problem, which is a result of Hayman [8, p60]:

Theorem 69 *Suppose that $f(z)$ is a function transcendental and meromorphic in the plane. Then either f assumes every finite value infinitely often in the plane, or each of its derivatives $f^{(k)}$, $k \geq 1$, assumes every finite non-zero value infinitely often in the plane.*

We ask what sets E exist such that this result continues to hold on $\mathbf{C} - E$, either for all transcendental meromorphic f or for some subset of such functions.

In this direction, Langley has proved the following two results.

For entire functions, we have [16]:

Theorem 70 *Given $\varepsilon > 0$, there exists $K(\varepsilon) > 0$, depending only on ε, such that if the sequence $\{a_n\}_{n=1}^{\infty}$ converges to infinity with*

$$|a_n - a_m| > \varepsilon |a_n|$$

for all $n \neq m$, while d_n satisfies

$$\log \frac{1}{d_n} > K(\varepsilon)(\log |a_n|)^2,$$

then for any polynomials $\alpha_0(z), \ldots, \alpha_k(z)$ (with $k \geq 1$ and $\alpha_k(z) \not\equiv 0$) and any transcendental entire function $f(z)$ such that

$$\psi(z) = \sum_{j=0}^{k} \alpha_j(z) f^{(j)}(z)$$

is non-constant, either $f(z)$ has infinitely many zeros outside

$$E = \cup_{n=1}^{\infty} B(a_n, d_n)$$

or $\psi(z)$ has infinitely many 1-points outside E.

Langley also exhibits an example to show that the spacing condition on the a_n is best possible.

For meromorphic functions, we have [13, p57]:

Theorem 71 *Suppose that $E = \{a_m\}_{m=1}^{\infty}$ satisfies*

$$\liminf_{m \to \infty} \frac{\log |a_{m+1}|}{\log |a_m| \log \log |a_m|} > 0.$$

Suppose that a, b are finite, with $b \neq 0$, and that f is meromorphic such that, in $\mathbf{C} - E$, f has only finitely many poles and a-points, and f' has only finitely many b-points. Then f is rational.

Langley also showed that this theorem does not hold if the point set E is replaced by any countable set of open discs tending to ∞.

We shall aim to improve Langley's result for meromorphic functions by removing the restriction on the position of the poles, and by improving the spacing of the points a_m, although our result will be more restricted in the sense that it will only apply to functions which have Nevanlinna deficient poles. We shall also prove a result for higher derivatives of f. We prove:

Theorem 72 *Let $k \geq 1$ be an integer. Let a and $b \neq 0$ be complex numbers. Suppose that δ, ν satisfy*

$$1 \geq \delta > 1 - 1/k, \tag{5.1}$$

$$\nu > \frac{1}{(1-k(1-\delta))^2} \geq 1. \tag{5.2}$$

Let $E = \{a_m\}_{m=1}^{\infty}$ be a set of points with $|a_1| > e$ and

$$|a_{m+1}| > |a_m|^{\nu} \tag{5.3}$$

for all m.

If f is a transcendental meromorphic function, with

$$\delta(\infty, f) \geq \delta, \tag{5.4}$$

then either $f(z) = a$ infinitely often, or $f^{(k)}(z) = b$ infinitely often, for $z \in \mathbf{C} - E$.

Note that, by considering the function $\frac{f(z)-a}{b}$, we may assume that $a = 0$, $b = 1$.

Also, (5.1) gives

$$0 \leq k(1-\delta) < 1, \quad 0 < 1 - k(1-\delta) \leq 1. \tag{5.5}$$

5.2 Results required in the proof

Lemma 73 (Barry [5]) *Suppose that* g, h *are positive non-decreasing real functions and suppose that*

$$\int_1^r h(t)\frac{dt}{t} \leq \int_1^r g(t)\frac{dt}{t} \tag{5.6}$$

for all large r.

Let $\Lambda > 1$ *and let* $S = \{r \geq 1; h(r) > \Lambda g(r)\}$.

Then the set S *has* upper logarithmic density *no greater than* $1/\Lambda$, *ie*

$$\limsup_{s\to\infty}(\log s)^{-1}\int_1^s \chi_S(r)\frac{dr}{r} \leq 1/\Lambda,$$

where χ_S *is the characteristic function of the set* S.

Lemma 74 (Langley [17]) *Let* $n(t)$ *be non-decreasing, integer-valued and continuous from the right such that* $n(1) = 0$ *and* $n(t) = o(\log t)$ *as* $t \to \infty$. *Set*

$$h(r) = \int_1^r t dn(t).$$

If μ *is a positive constant, then the set* $E(\mu) = \{r \geq 1; h(r) \geq \mu r\}$ *has upper logarithmic density* 0.

Lemma 75 (Miles/Rossi - see [17]) *Let* $m(t)$ *be non-decreasing, integer-valued and continuous from the right, with* $m(1) = 0$ *and* $m(t) = O(t)$ *as* $t \to \infty$. *Let* $M > 3$ *be a constant. Then there exists a set* E_M *of lower logarithmic density at least* $1 - 3/M$, *ie*

$$\int_1^r \chi(t)/t dt > (1 - 3/M + o(1))\log r$$

as $r \to \infty$, *with* $\chi(t)$ *the characteristic function of* E_M, *such that, for* $r \in E_M$ *and* $t \geq r$, *we have* $m(t)/m(r) \leq (t/r)^{4M}$.

5.3 Lemmas required in the proof

We shall require the following slight modification of a lemma of Langley ([17, Lemma 2]):

Lemma 76 *Let f be transcendental and meromorphic in the plane with $T(r,f) = o((\log r)^2)$. Let τ be a small positive constant. Then there is a set W of upper logarithmic density at most τ such that, for $r \notin W$, there exist non-zero $b(r) \in \mathbf{C}$ and $\lambda(r) \to \infty$ as $r \to \infty$ such that*

$$f(z) = b(r) z^{n(r,1/f)-n(r,f)}(1 + o(1)),$$

for $r/\lambda(r) < |z| < r\lambda(r)$.

We prove Lemma 76 using the same method as in [17].
We write $f(z) = U(z)F(z)$ where U is a rational function and $F(0) = 1$ and F has no zeros or poles in $|z| \leq 1$. We choose a small $\mu > 0$ and apply Lemma 74 with $n(t) = n(t,F) + n(t,1/F) = O(T(t^2,f)/\log t) = o(\log t)$. We also apply Lemma 75 with $M > 3/\tau$ and $m(t) = 2^{n(t)} - 1$. We obtain that, for r outside a set W of upper logarithmic density at most τ, we have

$$h(r) = \int_1^r t dn(t) < \mu r \tag{5.7}$$

and, for $t \geq r$,

$$n(t) - n(r) \leq M_1 \log(t/r), \tag{5.8}$$

for $M_1 = 4M/\log 2 + 1$.
Since $n(t)$ is integer-valued, (5.7) implies that f has no zeros or poles in $\mu r < |z| < r$, for $r \notin W$.

Now suppose that

$$r \notin W, \quad \mu^{3/4} r \leq |z| \leq \mu^{1/4} r. \tag{5.9}$$

We write $F(z) = f_1(z)/f_2(z)$ where the f_j are entire and

$$f_1(z) = \prod_{j=1}^{\infty}(1 - z/x_j),$$

where the x_j are the zeros of f in $|z| > 1$, counting multiplicities. For z as in (5.9) we have

$$f_1(z) = z^{n(r,1/F)} \prod_1 (-1/x_j) \prod_1 (1 - x_j/z) \prod_2 (1 - z/x_j),$$

where $\prod_1$ denotes the product over all x_j with $|x_j| < r$, and $\prod_2$ denotes the product over the remaining x_j. With $\sum_1$ defined analogously to $\prod_1$, we have, using (5.7),

$$\begin{aligned} |\prod_1 (1 - x_j/z) - 1| &\leq \exp(\sum_1 |x_j/z|) - 1 \\ &\leq \exp(h(r)/|z|) - 1 \leq \exp(\mu r/|z|) - 1 \\ &\leq \exp(\mu^{1/4}) - 1. \end{aligned}$$

Further, (5.8) gives

$$n(t, 1/f) - n(r, 1/f) \leq M_1 \log(t/r)$$

for $t \geq r$, and we have

$$\begin{aligned} |\prod_2 (1 - z/x_j) - 1| &\leq \exp(|z| \int_r^{\infty} \frac{1}{t} dn(t, 1/f)) - 1 \\ &= \exp(|z| \int_r^{\infty} (n(t, 1/f) - n(r, 1/f)) dt/t^2) - 1 \\ &\leq \exp(|z| M_1 \int_r^{\infty} \log(t/r) dt/t^2) - 1 \\ &= \exp(|z| M_1/r) - 1 \leq \exp(M_1 \mu^{1/4}) - 1. \end{aligned}$$

Now if $\varepsilon > 0$ is given, we may choose μ small enough that we obtain

$$f_1(z) = \prod_1 (-1/x_j) z^{n(r,1/F)} (1 + \xi(z))$$

with $|\xi(z)| < \varepsilon$ for z satisfying (5.9).

Estimating f_2 in the same way gives the result. Lemma 76 is proved.

5.4 Proof of Theorem 72

We suppose that there exist f and E satisfying the hypotheses, such that $f(z) = 0$ only finitely often outside E and $f^{(k)}(z) = 1$ only finitely often outside E. We shall aim for a contradiction, which will prove the theorem.

By Chen's improvement [7] to the inequality known as Hayman's alternative, we have

$$T(r, f) < (2 + \frac{1}{k})N^{(k+2)}(r, 1/f) + (2 + \frac{2}{k})\overline{N}(r, \frac{1}{f^{(k)} - 1}) + S(r, f), \quad (5.10)$$

where $N^{(j)}(r, 1/f)$ is the same as $N(r, 1/f)$ except that zeros of multiplicity greater than j are counted j times only.

We set

$$N_E(r) = \sum_{|a_m|<r} \log \frac{r}{|a_m|},$$

where here we only include terms in the summation if either $f(a_m) = 0$ or $f^{(k)}(a_m) = 1$. From (5.10), we have

$$T(r, f) < (2k + 11)N_E(r) + S(r, f). \quad (5.11)$$

If $|a_n| < r \leq |a_{n+1}|$, we have, since $|a_1| > e$,

$$N_E(r) \leq \log \frac{r^n}{|a_n||a_{n-1}| \ldots |a_1|} < n \log r. \quad (5.12)$$

But

$$r > |a_n| > |a_{n-1}|^{\nu} > \ldots > |a_1|^{\nu^{n-1}} \quad (5.13)$$

and so

$$\log\log r > (n-1)\log\nu + \log\log|a_1| > (n-1)\log\nu. \tag{5.14}$$

So

$$N_E(r) = O(\log r \log\log r) = o((\log r)^2) \tag{5.15}$$

and (5.11) becomes

$$T(r,f) < (2k+11)N_E(r) + S(r,f) = o((\log r)^2). \tag{5.16}$$

Since f has deficient poles and $T(r,f) = o((\log r)^2)$, we now deduce from Theorem 30 that

$$T(r,f) = (1+o(1))N(r,1/f). \tag{5.17}$$

Lemma 77 *We can choose positive constants c_j, $1 \leq j \leq 8$, with the following properties:*

$$1 - 1/k < c_2 < c_1 < \delta \tag{5.18}$$

$$\frac{1-c_1}{1-c_2} < \frac{c_3-c_4}{1-c_4} \tag{5.19}$$

$$\frac{1-c_1}{1-c_2} < \frac{c_6-c_5}{1+c_6} \tag{5.20}$$

$$c_3 < c_7 \leq \frac{c_8}{1+c_8} \tag{5.21}$$

$$c_6 < c_8 \tag{5.22}$$

$$\nu^{-1} \leq (\nu(1-k(1-\delta)))^{-1} < 1-c_7 < 1-c_3 < 1-c_4 < 1 <$$
$$1+c_5 < 1+c_6 < 1+c_8 < \nu(1-k(1-\delta)) \leq \nu. \tag{5.23}$$

From (5.2), we can choose c_7 so that

$$1 < \nu(1 - c_7)^2 < \nu(1 - k(1 - \delta))^2. \tag{5.24}$$

So then

$$\frac{1}{1 - k(1 - \delta)} < \frac{\nu(1 - c_7)^2}{1 - k(1 - \delta)} < \nu(1 - c_7) < \nu(1 - k(1 - \delta))$$

and we can choose c_8 such that

$$\frac{1}{1 - k(1 - \delta)} < 1 + c_8 < \nu(1 - c_7) < \nu(1 - k(1 - \delta)). \tag{5.25}$$

Furthermore, (5.24) implies that

$$\frac{\nu(1 - c_7) - 1}{\nu(1 - c_7)} = 1 - \frac{1}{\nu(1 - c_7)} > c_7$$

and so we can choose $1 + c_8$ in (5.25) so close to $\nu(1 - c_7)$ that

$$\frac{c_8}{1 + c_8} = \frac{(1 + c_8) - 1}{1 + c_8} > c_7. \tag{5.26}$$

So the right hand inequality of (5.21) holds.

From (5.24) and (5.25) we have necessarily

$$c_7 > k(1 - \delta) \tag{5.27}$$

$$c_8 > \frac{1}{1 - k(1 - \delta)} - 1 = \frac{k(1 - \delta)}{1 - k(1 - \delta)} \geq k(1 - \delta). \tag{5.28}$$

For any c_1, c_2 which satisfy (5.18) we have

$$\frac{1 - c_1}{1 - c_2} > \frac{1 - \delta}{1/k} = k(1 - \delta)$$

and so (by taking c_1 as close to δ and c_2 as close to $1 - 1/k$ as necessary), we can choose c_1, c_2 which satisfy (5.18) and also (using (5.27)),

$$k(1 - \delta) < \frac{1 - c_1}{1 - c_2} < c_7. \tag{5.29}$$

Next, we can choose c_3, c_6 so that

$$\frac{1-c_1}{1-c_2} < c_3 < c_7 \tag{5.30}$$

$$\frac{1-c_1}{1-c_2} < \frac{c_6}{1+c_6} < c_7 \leq \frac{c_8}{1+c_8}, \tag{5.31}$$

using (5.26) and (5.29). The left hand inequality of (5.21) holds, and (5.22) holds.

Now, using (5.30) and (5.31), we choose

$$c_4 < c_3, \quad c_5 < c_6 \tag{5.32}$$

positive but so small that

$$\frac{1-c_1}{1-c_2} < \frac{c_3-c_4}{1-c_4} < c_7$$
$$\frac{1-c_1}{1-c_2} < \frac{c_6-c_5}{1+c_6} < c_7.$$

So (5.19) and (5.20) hold.

Note that, necessarily, from (5.5), (5.22), (5.25), (5.30) and (5.32) we have that (5.23) holds.

Lemma 77 is proved.

Returning to the proof of the theorem, we set

$$A_m = \{z; |a_m|^{1-c_7} < |z| < |a_m|^{1+c_8}\}. \tag{5.33}$$

Note that, for any m,

$$|a_m|^{1+c_8} < |a_m|^{\nu(1-c_7)} < |a_{m+1}|^{1-c_7},$$

using (5.3) and (5.25). So the A_m do not intersect.

We denote by p_m the number of zeros of f at a_m and by y_m the number of poles of f in A_m.

Lemma 78 *There exist infinitely many m such that*

$$y_m < \frac{1}{k} p_m. \tag{5.34}$$

Suppose the lemma does not hold. Then for all sufficiently large r we have, from (5.17) and (5.33) (using w_j to denote poles of f),

$$\begin{aligned}
T(r,f) &= (1+o(1))N(r,1/f) \\
&= (1+o(1)) \sum_{|a_m|<r} p_m \log \frac{r}{|a_m|} + O(\log r) \\
&\leq (1+o(1))k \sum_{|a_m|<r} y_m \log \frac{r}{|a_m|} + O(\log r) \\
&= (1+o(1))k \sum_{|a_m|<r} \sum_{w_j \in A_m} (\log \frac{r}{|w_j|} + \log \frac{|w_j|}{|a_m|}) + O(\log r) \\
&\leq (1+o(1))kN(r,f) + (1+o(1))k \sum_{|a_m|<r} y_m c_8 \log |a_m| + O(\log r) \\
&\leq k(1-\delta+o(1))T(r,f) + (1+o(1))k \sum_{|a_m|<r} y_m c_8 \log |a_m|.
\end{aligned}$$

If we can show that, for some $\delta' > 0$,

$$kc_8 \sum_{|a_m|<r} y_m \log |a_m| < (1 - k(1-\delta) - \delta')T(r,f) \tag{5.35}$$

for a suitable sequence of $r \to \infty$, then we will have a contradiction, which will prove the lemma.

Set $r_n = |a_n|^\nu$. By (5.2) and (5.3),

$$|a_n| < r_n < |a_{n+1}|$$

and so

$$\sum_{|a_m|<r_n} y_m \le n(|a_n|^{1+c_8}, f) \le (\log \frac{r_n}{|a_n|^{1+c_8}})^{-1} N(r_n, f) \le \frac{1-\delta+o(1)}{(\nu-1-c_8)\log|a_n|} T(r_n, f).$$

So

$$kc_8 \sum_{|a_m|<r_n} y_m \log|a_m| \quad \le \quad \frac{k(1-\delta+o(1))c_8}{\nu-1-c_8} T(r_n, f)$$

By (5.35), we need to show that

$$k(1-\delta)\left(\frac{c_8}{\nu-1-c_8}+1\right) < 1.$$

From (5.25) we have $1 + c_8 < \nu(1 - k(1 - \delta))$, and so

$$\begin{aligned} k(1-\delta)\left(\frac{c_8}{\nu-1-c_8}+1\right) &= k(1-\delta)\frac{\nu-1}{\nu-1-c_8} \\ &< k(1-\delta)\frac{\nu-1}{\nu k(1-\delta)} \\ &= \frac{\nu-1}{\nu} \\ &< 1 \end{aligned}$$

as required. Lemma 78 is proved.

Lemma 79 *Let*

$$\Lambda = \frac{1-c_2}{1-c_1} > 1, \tag{5.36}$$

from (5.18).

Then the set

$$S = \{r \geq 1 : n(r,f) > (1-c_2)n(r,1/f)\}$$

has upper logarithmic density at most $1/\Lambda$.

We use Lemma 73 with

$$h(t) = n(t,f), \quad g(t) = (1-c_1)n(t,1/f).$$

By (5.4), (5.17) and (5.18), we have

$$\begin{aligned}
\int_1^r h(t)\frac{dt}{t} &= N(r,f) \\
&\leq (1-\delta+o(1))T(r,f) \\
&= (1-\delta+o(1))N(r,1/f) \\
&< (1-c_1)N(r,1/f) \\
&= \int_1^r g(t)\frac{dt}{t}
\end{aligned}$$

for all large r, so the hypothesis (5.6) of Lemma 73 is satisfied.

Now Lemma 73 gives the required result immediately. Lemma 79 is proved.

Lemma 80 *For m large, there exist R'_m and S'_m,*

$$|a_{m-1}| < |a_m|^{1-c_3} < R'_m < |a_m|^{1-c_4}, \quad |a_m|^{1+c_5} < S'_m < |a_m|^{1+c_6} < |a_{m+1}|, \tag{5.37}$$

such that, for $j = 1, \ldots, k$, we have that $|f(z)|$ and $|f^{(j)}(z)|$ are large on $|z| = R'_m$, $|z| = S'_m$, and

$$\frac{f^{(j)}(z)}{f(z)} = \frac{1+o(1)}{z^j} M(|z|)(M(|z|)-1)\ldots(M(|z|)-j+1), \quad |z| = R'_m, |z| = S'_m, \tag{5.38}$$

in which

$$M(r) = n(r, 1/f) - n(r, f), \quad M(R'_m) \to \infty, M(S'_m) \to \infty. \tag{5.39}$$

We have, by (5.19),

$$(\log(|a_m|^{1-c_4}))^{-1} \int_{|a_m|^{1-c_3}}^{|a_m|^{1-c_4}} \frac{dt}{t} = \frac{c_3 - c_4}{1 - c_4} > \frac{1-c_1}{1-c_2} = 1/\Lambda$$

So for sufficiently large m, the set S in Lemma 79 cannot contain the whole interval $(|a_m|^{1-c_3}, |a_m|^{1-c_4})$. A similar argument, using (5.20), shows that S cannot contain the whole of the interval $(|a_m|^{1+c_5}, |a_m|^{1+c_6})$. Hence there exist R_m, S_m, with

$$|a_m|^{1-c_3} < R_m < |a_m|^{1-c_4}, \quad |a_m|^{1+c_5} < S_m < |a_m|^{1+c_6}, \tag{5.40}$$

such that

$$n(R_m, f) \leq (1-c_2)n(R_m, 1/f), \quad n(S_m, f) \leq (1-c_2)n(S_m, 1/f). \tag{5.41}$$

Note that, by (5.3) and (5.23),

$$|a_m|^{1-c_3} > |a_{m-1}|, \quad |a_m|^{1+c_6} < |a_{m+1}|$$

and so

$$|a_{m-1}| < R_m < |a_m| < S_m < |a_{m+1}|.$$

We take $\tau > 0$ very small and apply Lemma 76 to obtain R'_m, S'_m which fall outside the exceptional set W and satisfy

$$R_m^{1-\varepsilon} < R'_m < R_m, \quad S_m^{1-\varepsilon} < S'_m < S_m \tag{5.42}$$

for some small $\varepsilon > \tau$.

We fix a small $\mu > 0$ and obtain, for $R'_m(1-\mu) < |z| < R'_m(1+\mu)$, that

$$f(z) = b(R'_m) z^{n(R'_m,1/f)-n(R'_m,f)}(1+o(1)). \tag{5.43}$$

Note that (5.41) gives

$$n(R'_m, f) \leq n(R_m, f) \leq (1-c_2)n(R_m, 1/f) = (1-c_2)n(R'_m, 1/f). \tag{5.44}$$

The inequality (5.44), together with (5.17), shows that $M(R'_m) \to \infty$ in (5.39). Further, (5.43) shows that, for

$$|z| = r \in (R'_m(1-\mu), R'_m(1+\mu)),$$

we have, writing $M(r,f) = \max\{|f(z)| : |z| = r\}$,

$$\begin{aligned} \log|f(z)| &> \log M(r,f) - o(1) \\ &> m(r,f) - o(1) \\ &> (\delta - o(1))T(r,f), \end{aligned} \tag{5.45}$$

so that $f(z)$ is large for such z.

An estimate analogous to (5.43) holds for

$$S'_m(1-\mu) < |z| < S'_m(1+\mu),$$

and hence $f(z)$ is large on this region also, by the same argument.

Fix small positive

$$\gamma \le \frac{\mu^k}{(k+1)!}.$$

For m sufficiently large we can rewrite (5.43) as

$$f(z) = b(R'_m)z^M(1+\zeta(z)), \quad R'_m(1-\mu) < |z| < R'_m(1+\mu) \tag{5.46}$$

where we write

$$M = M(R'_m) = n(R'_m, 1/f) - n(R'_m, f) \to \infty$$

from (5.44), and

$$|\zeta(z)| < \gamma. \tag{5.47}$$

For $j = 1, \dots, k$, we have

$$\begin{aligned} f^{(j)}(z) = bM(M-1)\dots(M-j+1)z^{M-j} \quad & \Big(1 + \zeta(z) + \tfrac{j}{M-j+1}\zeta'(z)z + \dots \\ & \dots + \tfrac{1}{M(M-1)\dots(M-j+1)}\zeta^{(j)}(z)z^j\Big), \\ R'_m(1-\mu) < |z| < R'_m(1+\mu) \quad & \end{aligned} \tag{5.48}$$

Using Cauchy's estimate (Theorem 40), we have that, for $1 \le q \le j$,

$$|\zeta^{(q)}(z)| \le \frac{q!\gamma}{(\mu R'_m)^q}, \quad |z| = R'_m.$$

So the sum of the terms in ζ and its derivatives in the bracketed term in (5.48) has modulus which is $O(\gamma)$, and (5.38) follows for $|z| = R'_m$. A similar argument applies for $|z| = S'_m$.

Since $|f(z)|$ is large on $|z| = R'_m$, $|z| = S'_m$, then (5.38) implies that $|f^{(j)}(z)|$ is large there also.

Lemma 80 is proved.

Lemma 81 *For all sufficiently large m satisfying (5.34), we have $f^{(k)}(a_m) \neq 0$, while a_m is a zero of f of multiplicity no greater than k, f has no poles in A_m, and*

$$n(r, f) \leq (1 - c_2)n(r, 1/f), \quad r \in (|a_m|^{1-c_7}, |a_m|^{1+c_8}). \tag{5.49}$$

Suppose that m is large and satisfies (5.34). From (5.34), we must have $p_m \geq 1$ and so $f(a_m) = 0$. Suppose that the zero has multiplicity greater than k, so that $f^{(k)}(a_m) = 0$. We consider the annulus

$$D_m = \{z : R'_m < |z| < S'_m\},$$

which contains a_m.

Since $|f^{(k)}(z)|$ is large on the boundary of D_m, by Lemma 80, Rouché's Theorem shows that $f^{(k)}$ must have the same number of zeros and 1-points in D_m. So $f^{(k)}$ has at least one 1-point in D_m. But all 1-points of $f^{(k)}$ are at the a_n, and so $f^{(k)}(a_m) = 1$, by (5.37), contradicting our assumption that $f^{(k)}(a_m) = 0$. So $f^{(k)}(a_m) \neq 0$ and the zero of f at a_m has multiplicity no greater than k.

So $p_m \leq k$ and at (5.34) we have

$$y_m < \frac{1}{k}p_m \leq 1$$

and so we conclude that $y_m = 0$ i.e. there are no poles of f in A_m.

Moreover, recalling (5.33), this implies that $n(r, f) \leq (1 - c_2)n(r, 1/f)$ holds either for all $r \in (|a_m|^{1-c_7}, |a_m|)$ or for no such r. But since (5.30) and (5.36) give

$$\frac{1}{\log |a_m|} \int_{|a_m|^{1-c_7}}^{|a_m|} dt/t = \frac{\log |a_m| - \log |a_m|^{1-c_7}}{\log |a_m|} = c_7 > 1/\Lambda,$$

we conclude from Lemma 79 that $n(r,f) \le (1-c_2)n(r,1/f)$ holds for all $r \in (|a_m|^{1-c_7}, |a_m|)$. Similarly,

$$\frac{1}{\log|a_m|^{1+c_8}} \int_{|a_m|}^{|a_m|^{1+c_8}} dt/t = \frac{\log|a_m|^{1+c_8} - \log|a_m|}{\log|a_m|^{1+c_8}} = \frac{c_8}{1+c_8} > 1/\Lambda,$$

using (5.31) and (5.36), and so $n(r,f) \le (1-c_2)n(r,1/f)$ holds for all $r \in (|a_m|, |a_m|^{1+c_8})$.

But now also,

$$n(|a_m|,f) \le n(|a_m|+1,f) \le (1-c_2)n(|a_m|+1,1/f) = (1-c_2)n(|a_m|,1/f).$$

Lemma 81 is proved.

For large m satisfying (5.34) we have, from Lemma 81,

$$M(|a_m|) = n(|a_m|,1/f) - n(|a_m|,f) \ge c_2 n(|a_m|,1/f) > 0. \tag{5.50}$$

Lemma 82 *Let K be a large positive constant. Given $\varepsilon_1 > 0$, we have, for all sufficiently large m satisfying (5.34),*

$$\frac{f'(z)}{f(z)} = (1+\tau(z))\frac{M}{z} + \frac{p_m}{z-a_m}, \quad K^{-k}|a_m| < |z| < K^k|a_m|, \tag{5.51}$$

where $M = M(K^{-k}|a_m|)$ is large and positive and

$$|\tau(z)| < \varepsilon_1,$$

and where we recall that p_m denotes the number of zeros of f at a_m.

We have, from (5.23), (5.40) and (5.42),

$$|a_m|^{1-c_7} < R'_m < |a_m| < S'_m < |a_m|^{1+c_8}$$

and so, using (5.33) and Lemma 81 we have

$$M(r) = M(K^{-k}|a_m|)(1 + o(1)), \quad R'_m \le r \le S'_m.$$

The function

$$\phi(z) = \frac{z}{M}\left(\frac{f'(z)}{f(z)} - \frac{p_m}{z - a_m}\right)$$

is analytic on $R'_m \le |z| \le S'_m$, and satisfies

$$\phi(z) = 1 + o(1)$$

on the boundary, by (5.37) and (5.38) with $j = 1$. Hence (5.51) follows from the maximum principle. Lemma 82 is proved.

Lemma 83 *Let ε be fixed, $0 < \varepsilon < 4^{-(k+1)}$.*

Suppose that h is a function analytic in the region

$$K^{-k}|a_m| < |z| < K^k|a_m|, \quad |z - a_m| > \frac{|a_m|}{2^k}, \tag{5.52}$$

with

$$\frac{h'(z)}{h(z)} = \frac{M}{z}(1 + \tau_1(z)) \tag{5.53}$$

in that region, where $M = M(|a_m|) = n(|a_m|, 1/f) - n(|a_m|, f)$, and

$$|\tau_1(z)| \leq \varepsilon.$$

Then, for $j = 1, \ldots, k$ we have

$$\frac{h^{(j)}(z)}{h(z)} = \frac{M^j}{z^j}(1 + \tau_j(z)) \tag{5.54}$$

for

$$K^{j-1-k}|a_m| < |z| < K^{k-j+1}|a_m|, \quad |z - a_m| > \frac{|a_m|}{2^{k-j+1}}, \tag{5.55}$$

where

$$|\tau_j(z)| \leq 4^{j-1}\varepsilon.$$

If, in addition, h is analytic in the whole of the region

$$K^{-k}|a_m| < |z| < K^k|a_m|, \tag{5.56}$$

and satisfies (5.53) there, then (5.54) holds in

$$K^{j-1-k}|a_m| < |z| < K^{k-j+1}|a_m|. \tag{5.57}$$

We prove the first part. The proof is by induction on j. When $j = 1$, the assertion is simply a reiteration of the hypotheses of the lemma, so there is nothing to prove.

Now suppose that the result holds for $j-1$. We have

$$\frac{h^{(j)}(z)}{h(z)} = \frac{d}{dz}(\frac{h^{(j-1)}(z)}{h(z)}) + \frac{h^{(j-1)}(z)}{h(z)}\frac{h'(z)}{h(z)}. \tag{5.58}$$

Using the inductive hypothesis,

$$\frac{h^{(j-1)}(z)}{h(z)}\frac{h'(z)}{h(z)} = \frac{M^{j-1}}{z^{j-1}}(1+\tau_{j-1}(z))\frac{M}{z}(1+\tau_1(z)) = \frac{M^j}{z^j}(1+\sigma(z)) \tag{5.59}$$

in the region

$$K^{j-2-k}|a_m| < |z| < K^{k-j+2}|a_m|, \quad |z-a_m| > \frac{|a_m|}{2^{k-j+2}}, \tag{5.60}$$

where

$$|\sigma(z)| \le |\tau_{j-1}(z)| + 2|\tau_1(z)| \le 2(4)^{j-2}\varepsilon.$$

For z in the region (5.60),

$$|\frac{h^{(j-1)}(z)}{h(z)}| \le 2\frac{M^{j-1}}{|z|^{j-1}}.$$

Using Cauchy's estimate,

$$|\frac{d}{dz}\frac{h^{(j-1)}(z)}{h(z)}| \le 2\frac{M^{j-1}}{|z|^{j-1}}\frac{1}{\rho},$$

where ρ is the distance from z to the boundary of the region (5.60).

For z satisfying (5.55), we have

$$\rho \ge \min\{K^{j-1-k}|a_m| - K^{j-2-k}|a_m|, K^{k-j+2}|a_m| - K^{k-j+1}|a_m|, \frac{|a_m|}{2^{k-j+1}} - \frac{|a_m|}{2^{k-j+2}}\} \ge \frac{|a_m|}{K^{k-j+2}}$$

and so, for such z,

$$|\frac{d}{dz}\frac{h^{(j-1)}(z)}{h(z)}| \le 2\frac{M^{j-1}}{|z|^{j-1}}\frac{K^{k-j+2}}{|a_m|} = o(1)\frac{M^j}{|z|^j} \tag{5.61}$$

using the fact that $M \to \infty$ as $m \to \infty$.

So, for m large enough, from (5.58), (5.59) and (5.61), for z satisfying (5.55),

$$\frac{h^{(j)}(z)}{h(z)} = \frac{M^j}{z^j}(1 + \tau_j(z))$$

with

$$|\tau_j(z)| \leq 4^{j-1}\varepsilon$$

as required. The proof of the second part is identical. Lemma 83 is proved.

Lemma 84 *For m large and satisfying (5.34), we have*

$$\frac{f^{(k)}(z)}{f(z)} = \frac{M^k}{z^k}(1 + \tau_k(z)) \tag{5.62}$$

for

$$K^{-1}|a_m| < |z| < K|a_m|, \quad |z - a_m| > \frac{|a_m|}{2}, \tag{5.63}$$

where

$$|\tau_k(z)| < \frac{1}{2}.$$

We apply Lemma 83, using $h = f$. By Lemma 81, f is analytic in the region (5.52). In that region, by Lemma 82, we have

$$\frac{f'(z)}{f(z)} = (1 + \tau(z))\frac{M}{z} + \frac{p_m}{z - a_m}.$$

But also, from (5.52) and Lemma 81,

$$\left|\frac{p_m}{z - a_m}\right| < \frac{2^k p_m}{|a_m|} \leq \frac{2^k k}{|a_m|} = o(1)\frac{M}{|z|}$$

and so the hypothesis at (5.53) is satisfied.

The result follows. Lemma 84 is proved.

Lemma 85 *For large m satisfying (5.34), we have $f^{(k)}(a_m) = 1$.*

We recall from (5.34) that $p_m > 0$ and so $f(a_m) = 0$.

By Lemma 84 and Rouché's theorem, $f^{(k)}/f$ has the same number of (zeros minus poles) in $|z - a_m| < |a_m|/2$ as does $\frac{M^k}{z^k}$ ie zero. But $f^{(k)}/f$ has p_m poles in $|z - a_m| < |a_m|/2$, all at a_m (recalling from Lemma 81 that $f^{(k)}(a_m) \neq 0$ and that f has no poles in A_m).
So $f^{(k)}/f$ has p_m zeros inside $|z - a_m| < |a_m|/2$. These must be zeros of $f^{(k)}$.

But now, using the fact that $|f^{(k)}|$ is large on $|z| = R'_m$ and $|z| = S'_m$, from Lemma 80, we can apply Rouché's theorem to $f^{(k)}$ to conclude that $f^{(k)}$ must have a 1-point in $R'_m < |z| < S'_m$. But this must be at a_m, ie $f^{(k)}(a_m) = 1$. Lemma 85 is proved.

We are now in a position to obtain a contradiction, which will prove the theorem.

We take a large m which satisfies (5.34). From Lemma 85, we have

$$f(a_m) = 0, \quad f^{(k)}(a_m) = 1. \tag{5.64}$$

We set

$$f(z) = (z - a_m)^{p_m} l(z) \tag{5.65}$$

where l is analytic, non-zero in A_m, using Lemma 81 again.

We have

$$\log|l(z)| = \log|f(z)| - p_m \log|z - a_m| \geq dT(|a_m|^{1-c_7}, f) - O(\log|a_m|)$$

for z on the boundary of A_m, using the Anderson/Clunie theorem (Theorem 36), and so, for any positive integer C, we have

$$\log|l(z)| > C \log|z| \tag{5.66}$$

for such z, provided m is large enough, using the fact that f is transcendental. By the maximum modulus principle, (5.66) holds for all z in A_m and, in particular, $|l(a_m)|$ is large.

In the case $p_m = k$ we have

$$f^{(k)}(z) = k!l(z) + B_0(z)$$

where B_0 is analytic in A_m and $B_0(a_m) = 0$. But now we have a contradiction immediately in this case, since $|l(a_m)|$ is large from the remark following (5.66), but $f^{(k)}(a_m) = 1$ from (5.64).

So we may suppose that $p_m < k$.

We apply Lemma 83 using $h = l$. Since l has no zero at or near a_m, we use the second part of the lemma to obtain that

$$\frac{l^{(k-p_m)}(z)}{l(z)} = \frac{M^{k-p_m}}{z^{k-p_m}}(1 + \tau_{k-p_m}), \quad K^{-p_m-1}|a_m| < |z| < K^{p_m+1}|a_m|.$$

Together with (5.66), this implies that

$$|l^{(k-p_m)}(z)| > \frac{1}{2} M^{k-p_m} |z|^{C-k+p_m} \tag{5.67}$$

in that region.

But, from (5.65),

$$f^{(k)}(z) = \frac{k!}{(k-p_m)!} l^{(k-p_m)}(z) + B(z) \tag{5.68}$$

where B is analytic in A_m and $B(a_m) = 0$.

But now we have a contradiction since $l^{(k-p_m)}(z)$ is large from (5.67), but also $f^{(k)}(a_m) = 1$ from (5.64).

Theorem 72 is proved.

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