Some results on the value distribution of meromorphic functions

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Contents

Abstract

In Chapter 1 we introduce many of the concepts and techniques, including Nevanlinna theory, referred to throughout the rest of the thesis. In Chapter 2 we extend a result of Langley and Shea [30] concerning the distribution of zeros of the logarithmic derivative of meromorphic functions to higher order logarithmic derivatives. Chapter 3 details an alternative formulation, avoiding reference to the multiplicity of poles, of a result due to Chuang concerning differential polynomials. In Chapter 4 we generalise a theorem of Bergweiler and Eremenko [8] concerning transcendental singularities of the the inverse of a meromorphic function. In Chapter 5 we generalise a result of Gordon to show that an unbounded analytic function on a quasidisk has a strong form of unboundedness there. Chapter 6 contains a proof of a result concerning the normality of families of analytic functions such that the composition of any of these functions with a fixed (meromorphic) outer factor has no fixpoints in a given domain.

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Chapter 1

Introduction

This chapter introduces the concepts and techniques used throughout the rest of the thesis. For the sake of brevity, proofs are omitted. Where results are not attributed, they are standard, and may be found in texts such as [1], or [22] for results in Nevanlinna theory.

1.1 Analytic and meromorphic functions

A complex-valued function f is *analytic* at a point $a \in \mathbb{C}$ if there exists an $r > 0$ such that f may be represented by a series of the form

$$f(z) = a_0 + a_1(z-a) + \frac{a_2}{2!}(z-a)^2 + \frac{a_3}{3!}(z-a)^3 + \dots$$

which is convergent for all values of z in $B(a, r) = \{z : |z - a| < r\}$. Functions satisfying this definition are referred to by many writers as *holomorphic*, though the series definition is equivalent to the usual definition in terms of complex differentiability. We say that f is analytic on a domain D if f is analytic at all points in D, and f is *entire* if f is analytic at every point in $\mathbb{C}$.

The idea of representing functions in series form is comprehensively dealt with by the following theorem:

Theorem 1.1.1 (Laurent) *Suppose $A = \{z \in \mathbb{C} : R < |z - a| < S\}$, where $a \in \mathbb{C}$ and $0 \leq R < S \leq +\infty$, and suppose $f : A \to \mathbb{C}$ is analytic. Then there exist constants $a_k \in \mathbb{C}$ $(k \in \mathbb{Z})$ such that*

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - a)^k$$

for all $z \in A$. The series converges locally uniformly on A.

We say f has an *isolated singularity* at a if there is some $S > 0$ such that f is analytic in $0 < |z - a| < S$, but $f(a)$ is not necessarily defined. There are three types of isolated singularity, the nature of any particular one being easy to determine from the form of the Laurent series.

(i) If the Laurent series expansion of f about a contains only terms of non-negative power in $(z-a)$, (i.e. $a_k = 0$ for all $k < 0$), the singularity is *removable*, and setting $f(a) = a_0 = \lim_{z \to a} f(z)$ makes f analytic in $|z - a| < S$.

(ii) If some, but finitely many, of the a_k for negative k are non-zero, then we say f has a *pole* at a of *multiplicity* p, where $-p$ is the least k such that $a_k \neq 0$. As z approaches a, $f(z) \to \infty$.

(iii) If there are infinitely many non-zero a_k for negative k, f has an *essential singularity*, and the limit of f at a does not exist.

Given a domain D, we say that f is *meromorphic* on D if, for every $a \in D$, either f is analytic or a is an isolated singularity which is a pole.

We introduce the *spherical metric*: we identify $\mathbb{C}$ with the plane $Z = 0$ lying in $\mathbb{R}^3$, and consider the unit sphere S centred at the origin (this is the *Riemann sphere*). Given a point z in the plane, we connect the point $(0,0,1)$ to z with a straight line, and let z^* be the point on $S\backslash\{(0,0,1)\}$ at which this line intersects S. We identify the point $(0,0,1)$ with a notional "point at infinity". Then, given two points z, w in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, we define the *spherical distance* between z and w to be the distance in the standard metric in $\mathbb{R}^3$ between z^* and w^*.

Hence, if z or $w = \infty$, we have a meaningful concept of the distance between a finite point in the plane and infinity. If both z and w are finite, the spherical metric is equivalent to the standard metric in the plane.

If we have a meromorphic function f and set $f(a) = \infty$ at every pole a, the resulting function, still meromorphic, is continuous in terms of the spherical metric. We say a function $f(z)$ is meromorphic at ∞ if $f(1/z)$ is meromorphic at 0.

We conclude this section by recalling a special subclass of meromorphic functions, the *Möbius transformations*. These are the functions $\sigma(z)$ of the form

$$\sigma(z) = \frac{az+b}{cz+d},$$

where a, b, c and d are all in $\mathbb{C}$ and $ad - bc \neq 0$. Any Möbius transformation is a composition of finitely many translations $z \to z + \alpha$, $\alpha \in \mathbb{C}\backslash\{0\}$, scalar multiplications $z \to \beta z$, $\beta \in \mathbb{C}$, and applications of the map $z \to 1/z$. Clearly, then, the inverse of a Möbius transformation is itself a Möbius transformation. It is also useful to know that the Möbius transformations map the set of all straight lines and circles in $\mathbb{C}^*$ into itself.

A particularly useful subset of the Möbius transformations is the set of *rota-*

tions of the sphere. Under such a rotation $\tau(z)$, which has the form

$$\tau(z) = \frac{az - b}{\overline{b}z + \overline{a}},$$

where a, $b \in \mathbb{C}$ and $|a|^2 + |b|^2 > 0$, the spherical distance between any two points in $\mathbb{C}^*$ is invariant.

1.2 Elementary Nevanlinna theory

If we have an entire function f, we can examine its rate of growth by considering the function

$$M(r, f) = \max_{z \in S(0,r)} |f(z)|,$$

where $S(0, r)$ is the circle centred on 0 with radius r. It is standard that $M(r, f)$ is an increasing function of r for non-constant analytic f, by the maximum principle. (This well-known principle states that if D is a bounded domain in $\mathbb{C}$ and f is analytic on D and continuous on ∂D, then there exists a point $w \in \partial D$ such that $|f(z)| \leq |f(w)|$ for all $z \in D$. In particular, if C is a circle contained in a simply connected domain D on which f is analytic, and $|f| \leq M$ on C, then $|f| \leq M$ inside C). We may define the *order* of an analytic function by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

For example, the function e^z has order 1 according to this definition, and the function e^{z^k}, $k \in \mathbb{N}$, has order k.

If we wish to consider the rate of growth of a meromorphic function, the function $M(r, f)$ is of no use because f may have poles, and $M(r, f)$ is not necessarily an increasing function of r. This problem is addressed by Nevanlinna theory, a comprehensive treatment of which may be found in [22]. We introduce the subject with the following theorem:

Theorem 1.2.1 (Jensen) *Let R be finite and positive and let f be meromorphic and not identically zero in $|z| \leq R$. Let the (finitely many) zeros and poles of f in $0 < |z| < R$ be denoted $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively, with repetition according to multiplicity in each case (i.e. if b is a pole of multiplicity p, it will appear p times in the list). Let $c \in \mathbb{C}$ and $d \in \mathbb{Z}$ be such that the first term in the Laurent expansion of f about 0 is cz^d (valid in some annulus $0 < |z| < S$). Then*

$$\log |c| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| d\phi + \sum_{j=1}^{m} \log \frac{|a_j|}{R} - \sum_{k=1}^{n} \log \frac{|b_k|}{R} - d \log R. \quad (1.1)$$

We retain the notation of Theorem 1.2.1 in the following discussion. Let $n(r, f)$ denote the number of poles of f in the region $|z| \leq r$, counting multiplicity, and let $\mu(t) = n(t, f) - n(0, f)$. Using the integration by parts formula for Riemann-Stieltjes integrals (see [4, Chapter 7]), we obtain

$$\begin{aligned} \sum_{k=1}^{n} \log \frac{R}{|b_k|} &= \int_0^R \log(R/t) d\mu(t) \\ &= -\int_0^R (n(t, f) - n(0, f)) d(\log(R/t)) \\ &= \int_0^R (n(t, f) - n(0, f)) dt/t. \end{aligned} \quad (1.2)$$

We introduce the *(integrated) counting function* or *Anzahlfunktion* $N(r, f)$, defined by

$$N(r, f) = \int_0^r (n(t, f) - n(0, f)) dt/t + n(0, f) \log r. \quad (1.3)$$

We also define, for $x > 0$,

$$\log^+ x = \max\{\log x, 0\},$$

noting that

$$\log x = \log^+ x - \log^+ 1/x. \quad (1.4)$$

The *proximity function* or *Schmiegungsfunktion* $m(r, f)$ is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi. \tag{1.5}$$

Substituting equations (1.2), (1.3), (1.4) and (1.5) into Jensen's formula (1.1), we obtain

$$\log |c| = m(R, f) + N(r, f) - m(r, 1/f) - N(r, 1/f). \tag{1.6}$$

The *Nevanlinna characteristic* is defined by

$$T(r, f) = N(r, f) + m(r, f),$$

and so we note that equation (1.6) becomes

$$\log |c| = T(R, f) - T(R, 1/f). \tag{1.7}$$

It is easy to see, by considering poles and the inequalities

$$\log^+ xy \leq \log^+ x + \log^+ y$$

and

$$\log^+(x + y) \leq \log^+(2 \max\{x, y\}) \leq \log^+ x + \log^+ y + \log 2$$

that, if f_1 and f_2 are meromorphic,

$$T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2)$$

and

$$T(r, f_1 + f_2) \leq T(r, f_1) + T(r, f_2) + O(1).$$

These results, combined with (1.7), lead us to the first major result in Nevanlinna theory:

Theorem 1.2.2 (Nevanlinna's first fundamental theorem) *Let f be meromorphic and non-constant, and let $a \in \mathbb{C}$. Then*

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1).$$

This is an equidistribution theorem which essentially states that f has no special affinity for any particular value. If f is meromorphic and non-constant in $\mathbb{C}$, then $T(r,f)$ is large when r is large (as we shall see shortly), so either f takes the value a very often (which makes $N(r,1/(f-a))$ large) or f is very close to a on part of the circle $S(0,r)$ (which makes $m(r,1/(f-a))$ large).

Take $f(z)=e^z$, for example. Then f is entire and zero-free, so $N(r,f)=N(r,1/f)=0$, and it is easy to see that $m(r,f)=m(r,1/f)=r/\pi$. Also, $m(r,1/(f-1))$ is small, but f takes the value 1 very often.

For brevity, we write

$$T\left(r,\frac{1}{f-a}\right)=T(r,a,f)=T(r,a),$$

with $T(r,f)=T(r,\infty,f)$, and corresponding notation applying to N, m and n.

It is important to note that $T(r,f)$ is non-decreasing, a consequence of the following:

Theorem 1.2.3 (Cartan, [22]) *Let f be a non-constant meromorphic function on $B(0,R)$, and let $r\in(0,R)$. Then*

$$T(r,f)=\log^+|f(0)|+\frac{1}{2\pi}\int_0^{2\pi}N(r,e^{is},f)ds,$$

with minor modifications if $f(0)=\infty$.

We are now in a position to give a meaningful definition of the *order*, $\rho(f)$, of a meromorphic function. We define

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r}$$

and the *lower order*, $\lambda(f)$, to be

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

For the most part, the lower the order, the more likely f is to be 'well-behaved'. The following theorem tells us that for entire f this definition of order agrees with the one given at the beginning of this section.

Theorem 1.2.4 *Let f be meromorphic and non-constant in $|z| \leq R$. If f has no poles in $\{z : |z| \leq R\}$ and $0 < r < R$, then*

$$T(r,f) \leq \log^+ M(r,f) \leq \left(\frac{R+r}{R-r}\right) T(R,f).$$

An even finer quantification of a function's rate of growth comes from studying its *type*. Let f be a meromorphic function of finite order ρ, and let

$$s = \limsup_{r\to\infty} \frac{T(r,f)}{r^\rho}.$$

Then we say f has *minimal type*, *mean type* or *maximal type* if $s = 0$, s is finite and positive or $s = \infty$, respectively.

The following inequality often proves useful.

Lemma 1.2.1 *Let f be meromorphic in $|z| \leq R$, and let $0 < r < R$. Then*

$$N(R,f) \geq n(r,f) \log R/r + n(0,f) \log r.$$

Nevanlinna's second fundamental theorem is considerably more far-reaching than the first. We do not give a proof, but refer the reader to [22] for an in-depth account. First, we mention an important result and introduce some notation.

Lemma 1.2.2 (Lemma of the logarithmic derivative) *Let f be meromorphic and non-constant in the plane. Then there are positive constants C_1 and C_2 such that*

$$m(r, f'/f) \leq C_1 \log r + C_2 \log T(r, f)$$

as r tends to infinity outside a set E of finite measure.

We denote by $S(r, f)$ any quantity which is insignificant compared to $T(r, f)$ in the sense that

$$S(r, f) = O(\log^+(rT(r, f)))$$

outside a set E of finite measure. Hence, Lemma 1.2.2 shows that

$$m(r, f'/f) = S(r, f).$$

Since we can write $f^{(k)}/f$ as

$$\frac{f^{(k)}}{f^{(k-1)}} \frac{f^{(k-1)}}{f^{(k-2)}} \cdots \frac{f'}{f},$$

we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$$

for $k \in \mathbb{N}$.

It may be shown that $T(r, f) = O(\log r)$ if and only if f is a rational function, and so if f is non-rational, then $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside E.

Theorem 1.2.5 (Nevanlinna's second fundamental theorem) *Let f be meromorphic in the plane, and let b_j, $j = 1, \ldots, s$, be finitely many distinct values in $\mathbb{C}^*$. Then*

$$\sum_{j=1}^{s} m(r, b_j, f) \leq 2T(r, f) - N_1(r, f) + S(r, f), \tag{1.8}$$

where $N_1(r,f)$ counts the multiple points of f. Alternatively, adding the $N(r,b_j,f)$ terms to each side of (1.8) and applying the first fundamental theorem gives

$$(s-2)T(r,f) \leq \sum_{j=1}^{s} N(r,b_j,f) - N_1(r,f) + S(r,f). \tag{1.9}$$

It is illustrative of the power of this result to show that Picard's theorem is an immediate corollary. Suppose f is transcendental and meromorphic in the plane, taking three distinct values b_j, $j \in \{1,2,3\}$, finitely often. Then $N(r,b_j,f) = O(\log r)$ for these b_j, and so (1.9) becomes

$$T(r,f) \leq O(\log r),$$

contradicting the result mentioned above, that $T(r,f) = O(\log r)$ if and only if f is rational.

1.3 Harmonic and subharmonic functions

The results in this section may be found in [34] and [26]. Let D be a domain in $\mathbb{C}$. A function $u : D \to \mathbb{R}$ is called *harmonic* if u has continuous first and second partial derivatives and satisfies the Laplace equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0.$$

The Cauchy-Riemann equations show that if $f = u + iv$ with u and v real and f analytic, then u and v are harmonic. Conversely, if we have a harmonic function u on a simply connected domain D, it is possible to find an analytic function f such that $u = \mathbf{Re}(f)$.

Also, if u is harmonic and g is analytic, the composition $u \circ g$ is (locally) the real part of an analytic function, and hence harmonic.

Theorem 1.3.1 (Identity theorem for harmonic functions) *Let u be harmonic on a domain D in $\mathbb{C}$, and suppose u is constant on a non-empty subdomain G of D. Then u is constant on D.*

All harmonic functions have the *mean value property* (MVP). If D is a domain in $\mathbb{C}$ and $u : D \to \mathbb{R}$ is integrable, u has the MVP if to each $z_0 \in D$ there corresponds an $r_0 > 0$ such that

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt$$

for $0 < r \leq r_0$.

We now turn to subharmonic functions, first introducing some terminology. Let D be a domain in $\mathbb{C}$, and let u be a function mapping D into $\mathbb{R} \cup \{-\infty\}$. We say u is *upper semi-continuous* (USC) in D if $u(z_0) < t$ implies the existence of a positive δ such that $u(z) < t$ on $B(z_0, \delta)$.

Further, we say that u has the *sub-mean value property* (SMVP) if to each z_0 in D there corresponds an $r_0 > 0$ such that

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt, \tag{1.10}$$

for $0 < r < r_0$.

A function $u : D \to [-\infty, \infty)$ is *subharmonic* if it is upper semi-continuous and has the SMVP. The integral in equation (1.10) always exists if u is USC, since in this case u is measurable and bounded above on the circle. Harmonic functions are obviously subharmonic. If f is analytic on a domain D, then the function

$$u(z) = \begin{cases} \log|f(z)| & \text{if } f(z) \neq 0 \\ -\infty & \text{if } f(z) = 0 \end{cases}$$

is also subharmonic in D. Also, for $p > 0$, the function $u(z) = |f(z)|^p$ is subharmonic in D. These two facts prove to be extremely useful in diverse contexts.

Further, if u and v are subharmonic, then so are $u + v$ and $\max\{u, v\}$.

The well-known maximum principle for analytic functions, which was mentioned in section 1.2, may be extended to cover subharmonic functions as follows.

Theorem 1.3.2 (The maximum principle) *Let D be a domain in $\mathbb{C}$ and define $\partial_\infty D$ to be the collection of all boundary points of D in $\mathbb{C}^*$, with respect to the spherical metric (that is, $\partial_\infty D$ is the finite boundary of D plus, if D is unbounded, the point ∞). Then $\partial_\infty D$ is compact in $\mathbb{C}^*$. If u is subharmonic in D and*

$$\limsup_{z \to \zeta, z \in D} u(z) \leq M \in [-\infty, \infty)$$

for every $\zeta \in \partial_\infty D$, then either $u(z) \equiv M$ on D, or $u(z) < M$ for all z in D.

We conclude this section with a powerful theorem.

Theorem 1.3.3 *Let u be subharmonic and bounded above in $\mathbb{C}$. Then u is constant.*

1.4 Harmonic measure

Let D be a domain in $\mathbb{C}$ and let f be a bounded real function on $X = \partial_\infty D$. The Dirichlet problem is to find, if possible, a function h harmonic on D such that for each $w \in X$ we have $h(z) \to f(w)$ as $z \to w$, $z \in D$. The problem is not always solvable, but if it is, then

$$h(z) = \sup\{u(z) : u \in U(f)\},$$

where $U(f)$ is the collection of all functions u subharmonic in D such that for every $w \in \partial_\infty D$ we have

$$\limsup_{z \to w, z \in D} u(z) \leq f(w).$$

We describe some types of domain for which the Dirichlet problem is always solvable. Let $x_0 \in \partial_\infty D$. A *barrier* for D at x_0 is a subharmonic function b defined on $D \cap N$, where N is a neighbourhood of x_0, such that $b(z) < 0$ for $z \in D \cap N$ and

$$\lim_{z \to x_0, z \in D \cap N} b(z) = 0.$$

If the barrier exists then x_0 is called a *regular* boundary point, and D is said to be regular if all its boundary points are regular. Note that simply connected proper subdomains of $\mathbb{C}$ are regular, as we can write $b(z) = \log|F(z)|$, where F is the Riemann mapping. In general, if $x_0 \in \partial D_\infty$, E is the component of ∂D_∞ containing x_0 and $E \neq \{x_0\}$, then x_0 is regular. See [34] for more details concerning the barrier as defined here. (The definition of 'barrier' found in [34] is less restrictive than some of the alternatives.)

We say D is semi-regular if $\partial_\infty D$ is infinite and all but finitely many $x \in \partial_\infty D$ are regular. If D is semi-regular and f is bounded on $\partial_\infty D$ and continuous at all but finitely many points of $\partial_\infty D$, we can use the method of solving the Dirichlet problem above to obtain a harmonic function $v_f(z)$ on D such that

$$\lim_{z \to x, z \in D} v_f(z) = f(x)$$

for all regular points x of $\partial_\infty D$.

Without delving too deeply into measure theory (see [34] for more details), we can fix a z in D and construct a probability measure $\mu_z(A)$ on $\partial_\infty D$ such that

$$\omega(z, A, D) = \mu_z(A)$$

is a harmonic function of z on D for every Borel subset A of $\partial_\infty D$. We call this the *harmonic measure* of A with respect to D, evaluated at z. One interpretation of the harmonic measure is that $\omega(z, A, D)$ gives the probability that a particle exhibiting Brownian motion and starting at z will eventually exit D via some point of A.

The following two theorems are needed later.

Theorem 1.4.1 (The 'two constants' theorem) *Let E_j be finitely many pairwise disjoint Borel subsets of X, with union X. Let u be a function subharmonic and bounded above on D, and let $M_j \in \mathbb{R}$ be such that*

$$\limsup_{z \to x, z \in D} u(z) \leq M_j,$$

for $x \in E_j$. Then

$$u(z) \leq \sum_j M_j \omega(z, E_j, D)$$

for $z \in D$.

Theorem 1.4.2 (Conformal invariance) *Let D_1 and D_2 be semi-regular domains in $\mathbb{C}$, and let $X_j = \partial_\infty D_j$. Let B_j be a Borel subset of X_j and let $f : D_1 \cup B_1 \to D_2 \cup B_2$ be continuous, such that $f : D_1 \to D_2$ is analytic and f maps B_1 into B_2. Suppose further that at most finitely many x in B_1 are such that $f(x)$ is an irregular boundary point of D_2. Then*

$$\omega(z, B_1, D_1) \leq \omega(f(z), B_2, D_2).$$

The formula usually referred to as Tsuji's estimate is a particularly enlightening result concerning harmonic measure. Let D be a semi-regular domain in $\mathbb{C}$ and let $z \in D$. We define D_r to be the component of $D \cap B(0,r)$ containing z, and $\theta_D^*(r)$ as follows: if D contains the whole circle $S(0,r)$, then $\theta_D^*(r) = \infty$, otherwise $D \cap S(0,r)$ consists of countably many open arcs, and we define $\theta_D^*(r)$ to be the angular measure of the longest of these (note that if one has angular measure $s > 0$, then at most finitely many can have angular measure greater than s).

Theorem 1.4.3 (Tsuji, [38]) *Let D be a semi-regular domain in $\mathbb{C}$, let $z \in D$, let $0 < r < \infty$ and let D_r and θ_D^* be defined as above. Suppose $0 < k < 1$ and $2|z| \le kr$. Then*

$$\omega(z, S(0,r), D_r) \le C \exp\left(-\pi \int_{2|z|}^{kr} \frac{dt}{t\theta_D^*(t)}\right).$$

Here C is a positive constant depending only on k.

Tsuji's estimate is useful for determining the rate of growth of subharmonic functions, as in the following application.

Theorem 1.4.4 *Let v be subharmonic on the semi-regular domain D in $\mathbb{C}$, and assume that*

$$\limsup_{z \to \zeta, z \in D} v(z) \le 0$$

for every finite boundary point ζ of D. Assume further that $r_n \to \infty$ and

$$B_D(r_n, v) \exp\left(-\pi \int_1^{r_n/2} \frac{dt}{t\theta_D^*(t)}\right) \to 0$$

as $n \to \infty$, where

$$B_D(r,v) = \sup\{v(z) : z \in D, |z| = r\}.$$

Then $v(z) \le 0$ on D.

1.5 The hyperbolic metric

The hyperbolic metric is discussed in detail in [2]. Let γ be a smooth contour in the unit disc $B(0,1)$. The *hyperbolic length* of γ is defined to be

$$L_\gamma = \int_\gamma \frac{2|dz|}{1-|z|^2},$$

where $|dz|$ indicates integration with respect to arc length. Some writers omit the factor 2.

It can be shown that under a conformal map f from $B(0,1)$ onto $B(0,1)$, the hyperbolic length of a curve is invariant, and also that if γ joins 0 to $r \in (0,1)$, then

$$L_\gamma \geq \int_0^r \frac{2dx}{1-x^2} = \log\left(\frac{1+r}{1-r}\right).$$

Thus, in terms of hyperbolic length, the shortest path from 0 to r is the straight line segment.

If z_1 and z_2 are points in $B(0,1)$, we define the hyperbolic distance between them, denoted $[z_1, z_2]$, to be the infimum of L_γ over all smooth contours γ joining z_1 to z_2. Since the distance is not altered if we apply a conformal map f of $B(0,1)$ onto itself, we can choose f so that $f(z_1) = 0$, $f(z_2) = r$ for some $r > 0$, so the shortest path between these two points is the straight line segment S from 0 to r. Hence, the shortest path from z_1 to z_2 is the arc $f^{-1}(S)$, which, since f is a Möbius transformation, is either a straight line through 0 or a circular arc which meets the circle $S(0,1)$ at right-angles.

Now, let D be a simply connected domain in $\mathbb{C}$, with $D \neq \mathbb{C}$. Then, by the Riemann mapping theorem, there exists an analytic function F mapping D conformally onto $B(0,1)$. We can thus define the hyperbolic distance between w_1 and w_2 in D to be the hyperbolic distance between $F(w_1)$ and $F(w_2)$ in

$B(0,1)$. This does not depend on which F we choose, because if G is another conformal map of D onto $B(0,1)$, then $F \circ G^{-1}$ is a conformal map of $B(0,1)$ onto itself, so that $[F(w_1), F(w_2)] = [G(w_1), G(w_2)]$.

The following theorem gives a useful lower bound for hyperbolic distance.

Theorem 1.5.1 *Let D be a simply connected domain in the finite plane, not containing the origin, and let w_1 and w_2 lie in D. For $t > 0$, let $\theta(t)$ equal the angular measure of the longest open arc of $S(0,t)$ which lies in D. Then*

$$[w_1, w_2]_D \geq \int_{|w_1|}^{|w_2|} \frac{dt}{t\theta(t)}.$$

1.6 Analytic continuation

The results in this section may be found in [2]. We first introduce the concept of *homotopy*. Let S be a path-connected topological space (so any two points a and b in S can be joined by a continuous $f : [0,1] \to S$ with $f(0) = a$ and $f(1) = b$). Let x_0 and x_1 be points in S, possibly the same, and let γ and σ be two paths in S, both defined on $[0,1] = I$ and such that $\gamma(0) = \sigma(0) = x_0$ and $\gamma(1) = \sigma(1) = x_1$.

If S is a disc, or $\mathbb{R}^n$, or a space that can be thought of as having no 'holes', then it is reasonable to suppose that we could continuously deform γ into σ via a family of paths in S. More formally, we say γ is *homotopic to σ in S* if there exists a continuous function $H(t,u) : I \times I \to S$ with the following properties:

(i) $H(0,u) = x_0$ and $H(1,u) = x_1$.

(ii) $H(t,0) = \gamma(t)$ and $H(t,1) = \sigma(t)$.

Note that the paths γ and σ may be defined on any closed interval $[a,b]$ rather than $[0,1]$ if this is more convenient. In this case $H(t,u)$ would of course be defined on $[a,b] \times I$. It is easy to see that homotopy is an equivalence relation.

For a path $\gamma : I \to S$, we define the *inverse path* γ^{-1} by $\gamma^{-1}(t) = \gamma(1-t)$. The path $\gamma'(t)$ is defined by

$$\gamma'(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \gamma^{-1}(2t-1) = \gamma(2-2t) & \frac{1}{2} \le t \le 1, \end{cases}$$

and we note that γ' is homotopic to the constant curve $\eta(t) = \gamma(0)$, $0 \le t \le 1$.

We say that the topological space S is *homotopy simply connected* (HSC) if every closed curve $\gamma : I \to S$ (i.e. $\gamma(0) = \gamma(1)$) is homotopic to the constant curve η defined by $\eta(t) = \gamma(0)$ for $0 \le t \le 1$. All convex domains in $\mathbb{C}$ are HSC, as is the Riemann sphere $\mathbb{C}^*$.

We now consider the problem of analytic continuation, beginning with an example. We define three functions, each with a corresponding domain on which they are analytic. The function $L_0(z) = \log z = \log|z| + i \arg z$, where the argument is chosen to lie in $(-\pi, \pi)$, is analytic in the domain $D_0 = \mathbb{C} \backslash \{x : x \le 0\}$; the function $L_1(z) = \log|z| + i \arg z$, with the argument chosen in the same way, is of course analytic on the upper half-plane $D_1 = \{z : \mathbf{Im}(z) > 0\}$; finally, the function $L_2(z) = \log|z| + i \arg z$, with the argument chosen to lie in $(0, 2\pi)$, is analytic on $D_2 = \mathbb{C} \backslash \{x : x \ge 0\}$.

Suppose we start at the point 1 and continue $L_0(z)$ counter-clockwise around the circle $S(0,1)$. The argument increases, and we observe that $L_0 = L_1$ in the quadrant $0 < \arg z < \pi/2$, and so we continue with L_1. The argument continues to increase, and we note that $L_1 = L_2$ in the quadrant $\pi/2 < \arg z <$

π, and we continue with L_2 all the way back round until we approach 1, where the argument tends to 2π. Thus we have 'continued' L_0 around the circle, although we have not returned to the original value.

We introduce some terminology: by a *function element*, we mean a pair (f, D), in which D is a domain in $\mathbb{C}^*$ and f is meromorphic on D.

Let (f, D) be a function element, let $z_0 \in D$ and let $\gamma : [a, b] \to \mathbb{C}^*$ be a path with $\gamma(a) = z_0$. Continuity is with respect to the spherical metric. An *analytic continuation* of (f, D) along γ is a family of function elements (f_t, D_t), $a \leq t \leq b$, with the following properties:

(i) $f_a = f$ on a neighbourhood of $z_0 = \gamma(a)$.

(ii) $\gamma(t) \in D_t$ for every $t \in [a, b]$.

(iii) For every $t \in [a, b]$ there exists $\rho_t > 0$ such that the following holds: for $a \leq s \leq b$, $|s - t| < \rho_t$, we have $\gamma(s) \in D_t$ and $f_s = f_t$ on a neighbourhood of $\gamma(s)$.

Strictly speaking, this is a *meromorphic* continuation, but we retain the dominant terminology. We say the analytic continuation is *finite-valued* if all the f_t map the corresponding D_t into $\mathbb{C}$ rather than $\mathbb{C}^*$.

If G and H are domains with $G \subseteq H$, we say that a function element (g, G) admits *unrestricted analytic continuation* (UAC) in H if (g, G) can be analytically continued along every path in H starting in G.

The next result shows that if two analytic continuations agree on part of a path γ, then they agree on the whole of γ.

Theorem 1.6.1 *Let (f, D) and (g, D) both be analytically continued along the path $\gamma : [a, b] \to \mathbb{C}^*$. If there exists $u \in [a, b] = J$ such that $f_u = g_u$ on a neighbourhood of $\gamma(u)$, then for every t in J we have $f_t = g_t$ on a neighbourhood of $\gamma(t)$.*

Before we describe the two main theorems of this section, we require some definitions. A *critical point* z of a meromorphic function f is a multiple point of f (i.e. a pole of multiplicity at least two or a point at which $f'(z) = 0$). A point z is critical if and only if there is no neighbourhood of z on which f is one-to-one. The *critical values* of f are the values taken by f at the critical points. For example, $\cos z$ has critical points $n\pi$, $n \in \mathbb{Z}$, and critical values ± 1.

An *asymptotic value* of f is an element w of $\mathbb{C}^*$ such that there exists a path γ tending to infinity such that f tends to w as z tends to infinity on γ. For example, e^z has asymptotic values 0 and ∞, as e^z tends to 0 or ∞ on any path γ such that $\mathbf{Re}(\gamma) \to -\infty$ or $+\infty$ respectively. Asymptotic values have no significance for rational functions, since f is transcendental if and only if infinity is an essential singularity.

Theorem 1.6.2 (Iversen) *Let f be transcendental and meromorphic in $\mathbb{C}$, and let a be a value taken by f at most finitely many times in $\mathbb{C}$. Then a is an asymptotic value of f.*

Note that, since we may regard an entire function f as a meromorphic function omitting the value infinity, infinity is an asymptotic value of every transcendental entire function. Since $\log |f|$ is subharmonic when f is entire, the definitive result in this direction is the following:

Theorem 1.6.3 (Lewis, Rossi, Weitsman [31]) *Let u be subharmonic in $\mathbb{C}$, let*

$$B(r,u) := \sup\{u(re^{i\theta}) : \theta \in [0,2\pi)\}$$

and suppose that

$$\lim_{r\to\infty} \frac{B(r,u)}{\log r} = +\infty.$$

Then there exists a path Γ tending to infinity with

$$\int_\Gamma e^{-\lambda u}|dz| < +\infty$$

for each $\lambda > 0$, and

$$\frac{u(z)}{\log|z|} \to +\infty$$

as $z \to \infty$ on Γ. Note that Γ is independent of λ.

The next two theorems deal with situations in which the inverse of a function admits UAC, and whether or not analytically continuing a function around a closed curve leads back to the same function element.

Theorem 1.6.4 *Let $f : \mathbb{C} \to \mathbb{C}^*$ be non-constant and meromorphic. Let $z_0 \in \mathbb{C}$ be a non-critical point of f, and let $\gamma : [0,1] \to \mathbb{C}^*$ be a path in $\mathbb{C}^*$ starting at $w_0 = f(z_0)$ and not meeting any critical or asymptotic value of f. Let g be that branch of the inverse function f^{-1} which is defined on a neighbourhood D of w_0 and maps w_0 to z_0. Then (g,D) admits analytic continuation along γ, with all the function elements g_t finite-valued.*

Theorem 1.6.5 (The monodromy theorem) *Let G be a domain in $\mathbb{C}^*$ and let (f,D) be a function element with $D \subseteq G$. Suppose that (f,D) admits UAC in G. Let z_0 and $z_1 \in D$, and let γ and σ be homotopic paths from z_0 to z_1 in G. Then analytic continuation of (f,D) along γ or σ leads to the same local function element near z_1.*

1.7 Normal families

We introduce the concept of *normality* for families of first analytic, then meromorphic, functions. See [37] for a detailed, modern account of the concepts of normality.

A family $\mathcal{F}$ of analytic functions on a domain $D \subseteq \mathbb{C}$ is *normal* in D if every sequence of functions (f_n) in $\mathcal{F}$ contains a subsequence which converges locally uniformly to either a limit function ϕ (which will itself be analytic) on D or to infinity on D.

The family $\mathcal{F}$ is said to be normal at a point z_0 in D if it is normal on some neighbourhood of z_0. A family of analytic functions is normal on a domain D if and only if it is normal at every point of D.

A family of functions $\mathcal{F}$ is said to be *locally bounded* on a domain D if to each z_0 in D there corresponds a positive number M and a radius δ (both possibly dependent on z_0) such that $|f(z)| \leq M$ for all $z \in B(z_0, \delta) \subseteq D$ and all $f \in \mathcal{F}$. Local boundedness implies normality.

Theorem 1.7.1 (Montel) *Let D be a domain in $\mathbb{C}$, and let $\mathcal{F}$ be a family of analytic functions on D which omit the two distinct fixed values a and $b \in \mathbb{C}$. Then $\mathcal{F}$ is normal.*

Schiff [37] refers to Montel's theorem as the *fundamental normality test.*

We turn to families of meromorphic functions, for which the concept of local boundedness is of no use. In its place we introduce *equicontinuity*: a family $\mathcal{G}$ on a domain D is said to be equicontinuous at a point $z' \in D$ if, for each

$\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, z') > 0$ such that

$$|g(z) - g(z')| < \varepsilon$$

whenever $z \in B(z', \delta)$, for all $g \in \mathcal{G}$. Naturally, we say $\mathcal{G}$ is equicontinuous on a set E if $\mathcal{G}$ is equicontinuous at every point of E. *Spherical equicontinuity* is defined similarly, except that all distances are calculated using the spherical metric.

Now, a family $\mathcal{G}$ of meromorphic functions in a domain D is said to be normal in D if every sequence of functions in $\mathcal{G}$ contains a subsequence which converges locally spherically uniformly to a limit function ψ. The limit function ψ will either be meromorphic or identically infinity. The family $\mathcal{G}$ is normal in D if and only if it is spherically equicontinuous in D.

Theorem 1.7.2 (Montel) *Let D be a domain in $\mathbb{C}$, and let $\mathcal{G}$ be a family of meromorphic functions which omit the three distinct fixed values a, b and $c \in \mathbb{C}^*$. Then $\mathcal{G}$ is normal on D.*

Clearly Theorem 1.7.1 is an immediate corollary of Theorem 1.7.2.

Theorem 1.7.3 (The Ahlfors five islands theorem) *Let D_j be simply connected domains in $\mathbb{C}^*$ with piecewise analytic boundaries and pairwise disjoint closures, for $j \in \{1, \ldots, 5\}$. Let $D \subseteq \mathbb{C}$ be a domain and denote by $\mathcal{F}_A = \mathcal{F}_A(D, \{D_j\}_{j=1}^5)$ the family of all meromorphic functions $f : D \to \mathbb{C}^*$ with the property that no subdomain of D is mapped conformally onto one of the domains D_j by f (such a subdomain is called a* simple island *over D_j). Then $\mathcal{F}_A$ is normal.*

This result is standard (see [22, Chapters 5 and 6]), but a proof using Nevanlinna theory can be found in [6].

There is a long-standing heuristic principle in complex analysis which states that in general a family of meromorphic functions with property $\mathcal{P}$ on a domain D will be normal if $\mathcal{P}$ reduces a function meromorphic in the plane to a constant. A good illustration is Montel's theorem; recall that Picard's theorem forces a function meromorphic in the plane which omits three distinct values to be constant.

This principle is usually attributed to a rather generous translation of a statement made by Bloch in [9] – *nihil est in infinito quod non prius fuerit in finito* – but in general is not true. A more rigorous formalisation due to Robinson and Zalcman is given in [39], many applications of which may be found in [40], although this is beyond the scope of this discussion. Nonetheless, the so-called Bloch principle provides the motivation for many problems in the study of normal families.

1.8 Geometric properties

1.8.1 Quasidisks and quasiconformal maps

We introduce quasiconformal maps with reference to [33]. Let J be a Jordan curve in $\mathbb{C}$. We say that J is a *quasicircle* if there exists an $M > 0$ such that

$$\text{diam } J(a,b) \leq M|a-b|$$

for all a, b in J, where $J(a,b)$ is the arc of J of smaller diameter between a and b. The inner domain of J is called a quasidisk. Alternatively, a piecewise smooth Jordan domain is a quasidisk if and only if it has no inward- or outward-pointing cusps (of angle 0 or 2π).

Note that if f is analytic on a domain D, we have

$$\frac{\partial f}{\partial \overline{z}} = 0$$

on D. A *K-quasiconformal* map ϕ on a domain D is permitted some limited dependence on $\overline{z}$, and satisfies

$$\left| \frac{\partial f}{\partial \overline{z}} \Big/ \frac{\partial f}{\partial z} \right| \leq \frac{K-1}{K+1} < 1$$

almost everywhere in D. A 1-quasiconformal map is fully conformal.

It may be shown that a function ϕ mapping $B(0,1)$ conformally onto quasidisk D may be extended to a map $\tilde{\phi}$, quasiconformal throughout the whole of $\mathbb{C}^*$ and taking values in $\mathbb{C}^*$. Further, if σ is any Möbius tranformation and τ is a Möbius transformation mapping $B(0,1)$ conformally onto itself, then the composition $\sigma \circ \tilde{\phi} \circ \tau$ is also quasiconformal throughout $\mathbb{C}$.

We conclude this section with a useful theorem, proved in [3, p.51].

Theorem 1.8.1 *Let D be a simply connected domain, let $K \geq 1$ be a real number and let a_1 and a_2 be points in D and let b_1 and b_2 be in $\mathbb{C}$. Then the family of K-quasiconformal maps ϕ such that $\phi(a_1) = b_1$ and $\phi(a_2) = b_2$ is equicontinuous, hence normal, and compact on D.*

1.8.2 Starlike domains and starlike maps

The information in this section is mostly taken from [13]. A *starlike* domain D is a domain with a *star-centre*, i.e. a point $w \in D$ such that, for all $z \in D$, the straight line segment joining z to w is wholly contained within D.

Suppose we are given a starlike domain D and a conformal map $f : \Delta \to D$ subject to the usual normalisation conditions (i.e. $f(0) = 0$ and $f'(0) = 1$). We say f is a *starlike map*. It can be shown that not only is $D = f(\Delta)$ starlike, but so is each $f(B(0, \rho))$, where $0 < \rho < 1$. Hence it follows that $\arg f(z)$ increases as z moves around the circle $|z| = \rho$ in the positive direction. That is,

$$\frac{\partial}{\partial\theta}(\arg f(\rho e^{i\theta})) \geq 0.$$

However,

$$\begin{aligned}\frac{\partial}{\partial\theta}(\arg f(\rho e^{i\theta})) &= \mathbf{Im}\left(\frac{\partial}{\partial\theta}\log f(\rho e^{i\theta})\right)\\ &= \mathbf{Im}\left(\frac{izf'(z)}{f(z)}\right)\\ &= \mathbf{Re}\left(\frac{zf'(z)}{f(z)}\right),\end{aligned}$$

where $z = \rho e^{i\theta}$, $\rho < 1$. This condition on the real part of $zf'(z)/f(z)$ is often a useful characterisation of starlike maps.

This idea is extended in the case of *strongly starlike* domains. While the values taken by $zf'(z)/f(z)$ for a starlike map f are restricted to the right half-plane, the (normalised) conformal map g associated with a strongly starlike domain satisfies the more restrictive condition

$$\left|\arg\left(\frac{zg'(z)}{g(z)}\right)\right| < \frac{\pi\alpha}{2}$$

for some $0 \leq \alpha < 1$.

Fait, Krzyż and Zygmunt prove in [15] that strongly starlike domains are examples of quasidisks, and explicitly give the construction of the relevant quasiconformal map.

1.9 Miscellaneous topics

1.9.1 The length-area principle

Let f be analytic on an open set D. We define $n(w)$ to be the number of roots in D of the equation $f(z) = w$. We introduce the length-area principle and refer the interested reader to [25, p. 29] for a more complete discussion.

Theorem 1.9.1 (The length-area principle) *Suppose $f(z)$ is analytic in an open set D, and that $p(R) = p(R, D)$ is defined according to*

$$p(R) = \frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\theta})d\theta,$$

with $n(w)$ as above. Let $l(R)$ equal the total length of the curves in D on which $|f(z)| = R$ and suppose that the area A of D is finite. Then

$$\int_0^{2\pi} \frac{l(R)^2}{p(R)}\frac{dR}{R} \leq 2\pi A,$$

where the integrand is taken to be zero if $l(R) = 0$ or $p(R) = +\infty$. In particular, $l(R) < +\infty$ for almost all R for which $p(R) < +\infty$.

1.9.2 The Lagrange interpolation formula

The following formula gives a useful way of expressing a polynomial approximation of a function, and we make use of it in chapter 5.

Let f be the function we wish to approximate, and let $x_0, \ldots, x_n$ be $n+1$ points at which we know the value of f, the corresponding values being denoted $f_0, \ldots, f_n$. Suppose the polynomial P_n, of degree at most n, defined by

$$P_n(x) = a_0 + a_1 x + \ldots + a_n x^n$$

for some appropriate coefficients a_n, approximates f, and is such that

$$P_n(x_j) = f(x_j) = f_j,$$

$0 \leq j \leq n$. Now, for $0 \leq k \leq n$, define L_k as follows:

$$L_k(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)}.$$

Since $L_k(x)$ is a polynomial of degree n which satisfies $L_k(x_j) = 0$ for $0 \leq j \leq n$, $j \neq k$, and $L_k(x_k) = 1$, then

$$Q_n(x) = \sum_{k=0}^{n} L_k(x) f_k$$

is a polynomial of degree n which satisfies $Q_n(x_j) = f_j$, $0 \leq j \leq n$, and hence $Q_n \equiv P_n$.

1.9.3 Rouché's theorem

Rouché's theorem is extremely useful in situations in which we need to find zeros and poles of functions in particular domains. We make extensive use of it in Chapter 6.

Theorem 1.9.2 (Rouché) *Let f and g be meromorphic on a domain D, and let C be a simple closed curve such that C and its interior lie in D. Suppose $|f| > |g|$ at all points on C. Then, denoting the number of poles and zeros inside C of a function f by P_f and Z_f respectively,*

$$Z_{f+g} - P_{f+g} = Z_f - P_f.$$

1.9.4 The Denjoy-Carleman-Ahlfors theorem

Suppose that f is transcendental and meromorphic in the plane, and that $f(z)$ tends to the finite complex number a as z tends to infinity along a path γ. i.e. a is an asymptotic value of f. Then we say that the inverse function f^{-1} has a *transcendental singularity over* a [32]. For each positive t, a domain $C(t)$ is uniquely determined as that component of the set $C'(t) = \{z \in \mathbb{C} : |f(z) - a| < t\}$ which has unbounded intersection with the path γ.

The singularity of f^{-1} over a corresponding to a path γ is said to be *direct* if $C(t)$, for some positive t, contains finitely many zeros of $f(z) - a$, and *indirect* otherwise. Clearly, if the singularity is direct then $C(t)$, for sufficiently small t, contains no zeros of $f(z) - a$. Singularities over infinity are classified in an obviously analogous way.

For example, $(\sin z)/z$ tends to zero as z tends to infinity along the positive real axis, and this singularity is indirect. On the other hand, e^z has direct singularities over 0 and infinity. Indeed, Iversen's theorem (Theorem 1.6.2) shows that every entire function has a direct transcendental singularity over infinity.

The Denjoy-Carleman-Ahlfors theorem relates to transcendental singularities, and in combination with the Bergweiler-Eremenko theorem (see chapter 4) and other results concerning asymptotic paths such as Lewis, Rossi and Weitsman's theorem (Theorem 1.6.3), provides a platform from which very general results may be obtained. See chapter 4 for examples of this, and see [24] for a complete discussion and proof of the theorem.

Theorem 1.9.3 (Denjoy-Carleman-Ahlfors) *Suppose that f is transcen-*

dental and meromorphic in the plane, and that the inverse function f^{-1} has $n \geq 2$ direct transcendental singularities lying over $a_1, \ldots, a_n$ (not necessarily distinct). Then the lower order of f is at least $n/2$.

Chapter 2

Zeros of higher order logarithmic derivatives

2.1 Introduction

For a meromorphic function f, the zeros of the logarithmic derivative f'/f are those zeros of f' at which f is non-zero. Clunie, Eremenko and Rossi prove in [11] that if f is transcendental and meromorphic of at most order $\frac{1}{2}$, minimal type, then f'/f has infinitely many zeros, and further that if f is transcendental and entire of at most order 1, minimal type, then f'/f has infinitely many zeros. These bounds are sharp; for example, e^z is entire with order 1 but mean type, and its logarithmic derivative is identically 1, and $\tan^2 \sqrt{z}$ is meromorphic with order $\frac{1}{2}$ and mean type, but its logarithmic derivative omits the value 0.

It is natural to attempt to generalise these results to $f^{(k)}/f$, where $k \geq 2$. However, this situation is more complicated, because while a zero of f'/f

must correspond to a zero of f' at which f is non-zero, a zero of $f^{(k)}/f$, for $k \geq 2$, need not exclude any value of f. Langley considers the problem of higher order logarithmic derivatives in [28], obtaining:

Theorem 2.1.1 *Suppose that k is a positive integer and f is meromorphic of finite order in the plane such that $f^{(k)}/f$ is transcendental of order less than $\frac{1}{2}$. Then $f^{(k)}/f$ has infinitely many zeros. The same conclusion holds if f has only finitely many poles and*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0. \tag{2.1}$$

Thus far, we have only considered results pertaining to the number of zeros, not their frequency. For the $k = 1$ case, Langley and Shea prove the following in [30]:

Theorem 2.1.2 *Suppose d_1 is an odd positive integer and that f is a transcendental meromorphic function with $T(r,f) = O(r^{d_1})$ as $r \to \infty$, and that f satisfies (2.1). Then either*

$$\liminf_{r\to\infty} \left(T(r,f) - \frac{r^{1/2}}{2} \int_r^\infty \frac{N(t,f)}{t^{3/2}} dt \right) < \infty$$

or

$$\liminf_{r\to\infty} \left(T(r,f'/f) - \frac{r^{1/2}}{2} \int_r^\infty \frac{N(t,f/f')}{t^{3/2}} dt - \frac{5+d_1}{2} \log r \right) < \infty.$$

We note that, for transcendental f, (2.1) ensures that f'/f is transcendental, and that the conclusions show that either f has a lot of poles, or f'/f has a lot of zeros. Combining this result with ideas from the proof of Theorem 1 of [28], we generalise this result for higher derivatives of f.

Theorem 2.1.3 [1] *With f and d_1 as in Theorem 2.1.2, in particular such that f satisfies (2.1), let $k \in \mathbb{N}$ be such that $f^{(k)}/f$ is transcendental. Then either*

$$\liminf_{r\to\infty}\left(T(r,f)-\frac{r^{1/2}}{2}\int_r^\infty \frac{N(t,f)}{t^{3/2}}dt-(k-1)\log r\right)<\infty \tag{2.2}$$

or

$$\liminf_{r\to\infty}\left(T(r,f^{(k)}/f)-\frac{r^{1/2}}{2}\int_r^\infty \frac{N(t,f/f^{(k)})}{t^{3/2}}dt-\frac{5+d_1+4k}{2}\log r\right)<\infty. \tag{2.3}$$

Note that we must explicitly assume $f^{(k)}/f$ to be transcendental, whereas (2.1) guarantees f'/f to be transcendental.

The coefficient of the log term in (2.3) in the case $k=1$ is not quite the same as in the original theorem, but since this term is insignificant compared to $T(r,f)$ for transcendental functions f, this is in practice no weaker a conclusion, and Theorem 2.1.3 can be regarded as a true generalisation of Theorem 2.1.2. It is perhaps also worth noting that the proof of Theorem 2.1.3 obviates the need to employ a rather complex lemma with multiple conclusions which was essential in the original proof of Theorem 2.1.2.

Before proving Theorem 2.1.3 in section 2.3, we discuss its implications and introduce some preliminary lemmas. The following corollary perhaps elucidates the implications of Theorem 2.1.3 more readily.

Corollary 2.1.1 *Suppose that f is transcendental meromorphic in the plane, of finite order, with lower order $\mu < 1$. If $f^{(k)}/f$ has lower order at least λ, then $N(r,f)+N(r,f/f^{(k)})$ has order at least* $\min\{\lambda,\frac{1}{2}\}$.

[1] A paper containing the proof of Theorem 2.1.3 has been submitted by the author for publication in the journal *Complex Variables*.

Proof Since $\mu < 1$,

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

Suppose the corollary is false. Then both $N(r,f)$ and $N(r,f/f^{(k)})$ have order less than $l < \min\{\lambda, \frac{1}{2}\}$. Considering $N(r,f)$ first, we have

$$\int_r^\infty \frac{N(t,f)}{t^{3/2}} dt < \int_r^\infty c_1 t^{l-3/2} dt < c_2 r^{l-1/2}$$

for some positive constants c_1 and c_2. Hence

$$r^{1/2} \int_r^\infty \frac{N(t,f)}{t^{3/2}} dt = O(r^l)$$

as $r \to \infty$. It is straightforward to show that $\mu \geq \lambda$, and so

$$T(r,f) - \frac{r^{1/2}}{2} \int_r^\infty \frac{N(t,f)}{t^{3/2}} dt > r^{\lambda - o(1)}$$

as $r \to \infty$. A similar argument shows that

$$T(r,f^{(k)}/f) - \frac{r^{1/2}}{2} \int_r^\infty \frac{N(t,f/f^{(k)})}{t^{3/2}} dt > r^{\lambda - o(1)}$$

as $r \to \infty$ also, but this contradicts the fact that at least one of (2.2) or (2.3) must be true.

2.2 Preliminaries

We need the following four lemmas.

Lemma 2.2.1 (Hayman, [23]) *Suppose that f is transcendental and meromorphic in the plane such that*

$$m(r,f) - r^{1/2} \int_r^\infty \frac{n(t,f)}{t^{3/2}} dt \to \infty \tag{2.4}$$

as $r \to \infty$, and suppose that $|f(z)| \leq M < \infty$ on a path Γ tending to infinity. Then there exists a function v, non-constant, non-negative and subharmonic *in the plane, such that* $v(z) \leq \log^+ |f(z)| + O(1)$ *for all* z. *Property (2.4) suffices to ensure that* ∞ *is an asymptotic value of* f.

Lemma 2.2.2 (Langley and Shea, [30]) *Let* f *be transcendental and meromorphic in the plane, and suppose that* $d \geq 1$ *is such that* $T(r,f) = O(r^d)$ *for all large* r. *Denote by* $L(r,R,f)$ *the total length of the level curves* $|f(z)| = R$ *lying in the region* $|z| < r$. *Then there exist arbitrarily large positive* R *such that* $f(z)$ *has no critical values on the circle* $S(0,R) = \{z : |z| = R\}$ *and*

$$L(r,R,f) = O(r^{d_2})$$

as $r \to \infty$, *where* $d_2 = (3+d)/2$.

Lemma 2.2.3 *Let* v *be a non-constant subharmonic function in the plane, and let* Γ *be a path in* $\mathbb{C}$ *tending to infinity such that* $v \leq T$ *on* Γ *for some positive constant* T. *Let* $B(r,v)$ *be defined according to*

$$B(r,v) := \sup\{v(re^{i\theta}) : \theta \in [0,2\pi)\}.$$

Then

$$\frac{B(r,v)}{\log r} \to \infty$$

as $r \to \infty$.

Lemma 2.2.4 *Let* v_1 *and* v_2 *be non-constant subharmonic functions in the plane, let* s_1 *and* s_2 *be real constants, and let* U_1 *and* U_2 *be disjoint, unbounded domains such that* $v_j \leq s_j$ *on* ∂U_j *and* $v_j(z_j) > s_j$ *for at least one point* z_j *in* U_j. *Then*

$$\log B(r,v_1) + \log B(r,v_2) \geq 2\log r - O(1)$$

for all large r, *where* $B(r,v)$ *is defined as in Lemma 2.2.3.*

Both Lemmas 2.2.3 and 2.2.4 can be proved by a standard application of Tsuji's estimate for harmonic measure (see Theorems 1.4.3 and 1.4.4 and also [30]).

Lemma 2.2.5 *Let f be transcendental and meromorphic such that $T(r,f) = O(r^{d_1})$ as $r \to \infty$ for some odd positive integer d_1. Let $k \in \mathbb{N}$, set*

$$h(z) = f(z)/f^{(k)}(z), \tag{2.5}$$

and assume h is transcendental. Let

$$h_1(z) = h(z)/z^{N_1} \quad \text{and} \quad f_1(z) = \frac{f(z)}{z^{k-1}}, \tag{2.6}$$

where $N_1 = d_2 + 2k + 1$ and $d_2 = (3 + d_1)/2$ is the constant associated with d_1 via Lemma 2.2.2. Suppose further that the constant $T > 0$ is such that $L(r, T, h_1) = O(r^{d_2})$ as $r \to \infty$ and there are no critical values of h_1 on $S(0, T)$ (arbitrarily large such T exist by Lemma 2.2), and that C is an unbounded component of the set $\{z : |h_1(z)| > T\}$ such that $C \subseteq \{z : |z| \geq R\}$, for some $R > 1$. Then $f_1(z) = O(1)$ and $f(z) = O(|z|^{k-1})$ as $z \to \infty$ in C.

Proof The proof is based on part of the proof of Theorem 1 in [28]. We partition the boundary ∂C of C into its intersections with the annuli $A_m = \{z : 2^{m-1} \leq |z| < 2^m\}$. On ∂C, we have $|h_1| = T$, so on $A_m \cap \partial C$, we have $|h| \geq T(2^{m-1})^{N_1}$, and hence $|h|^{-1} \leq c2^{-N_1(m-1)}$, using c here and henceforth to denote a positive constant, not necessarily the same at each occurrence. Now,

$$\begin{aligned} \int_{A_m \cap \partial C} |t|^{2k} |h(t)|^{-1} |dt| &\leq cL(2^m, T, h_1)(2^m)^{2k} 2^{-N_1(m-1)} \\ &\leq c(2^m)^{d_2} 2^{2km} 2^{-N_1(m-1)} \end{aligned}$$

by assumption, and so

$$\begin{aligned} \int_{\partial C} |t|^{2k} |h(t)|^{-1} |dt| &\leq \sum_{m=1}^{\infty} c(2^m)^{d_2} 2^{2km} 2^{-N_1(m-1)} \\ &= c2^{N_1} \sum_{m=1}^{\infty} 2^{-m} \\ &< \infty. \end{aligned} \tag{2.7}$$

Fix a point z_0 in C. Any point ζ in the closure $\overline{C}$ of C can be joined to z_0 by a path μ_ζ consisting of part of the circle $|z| = |z_0|$ and part of the ray $\arg z = \arg\zeta$. Replacing any part of μ_ζ which leaves $\overline{C}$ with an arc of ∂C gives $|h_1| \geq T$ on μ_ζ, and we now have

$$\int_{\mu_\zeta} |t|^{2k}|h(t)|^{-1}|dt| \leq c < \infty. \tag{2.8}$$

To see this, consider the contributions made to the integral by the part μ_1 of the ray, the part μ_2 of the circle, and the arcs μ_3 of ∂C, separately. We have

$$\int_{\mu_1} |t|^{2k}|h(t)|^{-1}|dt| \leq c\int_{|z_0|}^{\infty} s^{2k}s^{-N_1}ds \leq c$$

for μ_1,

$$\int_{\mu_2} |t|^{2k}|h(t)|^{-1}|dt| \leq 2\pi\frac{|z_0|^{2k+1-N_1}}{T} \leq c$$

for μ_2, and finally

$$\int_{\mu_3} |t|^{2k}|h(t)|^{-1}|dt| \leq c$$

for μ_3, using equation (2.7). Note that the constants are independent of ζ. We now set

$$v(z) = \int_{z_0}^{z} \frac{(z-t)^{k-1}}{(k-1)!}f^{(k)}(t)dt - f(z),$$

and note that, by expanding out the $(z-t)^{k-1}$ term, it can be seen that v admits unrestricted analytic continuation in C. We differentiate to obtain

$$v'(z) = \int_{z_0}^{z} \frac{(z-t)^{k-2}}{(k-2)!}f^{(k)}(t)dt - f'(z),$$

and so on, up to

$$v^{(k-1)}(z) = \int_{z_0}^{z} f^{(k)}(t)dt - f^{(k-1)}(z)$$

and

$$v^{(k)}(z) = f^{(k)}(z) - f^{(k)}(z) \equiv 0.$$

Hence $v(z)$ is a polynomial $P(z)$ of degree at most $k-1$. Now, near z_0,

$$f(z) + P(z) = \int_{z_0}^{z} \frac{(z-t)^{k-1}}{(k-1)!}f^{(k)}(t)dt = u(z), \tag{2.9}$$

i.e. $f + P - u \equiv 0$ near z_0.

Now take any ζ in $\overline{C}$, and suppose we have a path γ joining z_0 to ζ in $\overline{C}$. Consider the function $f + P - u$, which is identically zero on a neighbourhood of z_0, and can be analytically continued along γ to ζ. Since f and P are both single-valued, so too is u, and equation (2.9) holds throughout $\overline{C}$.

We now estimate the function f on C. First note that since $C \subseteq \{z : |z| \geq R > 1\}$ and the degree of P is at most $k-1$, we have

$$\left|\frac{P(z)}{z^{k-1}}\right| \leq c$$

on C. Now, for $\zeta \in \overline{C}$, we parametrize μ_ζ with respect to arc length s. Since $f_1(z) = f(z)/z^{k-1}$, it follows from (2.9) that

$$|f_1(\mu_\zeta(s))| \leq c + \int_0^s \left|\frac{(\mu_\zeta(s) - \mu_\zeta(t))^{k-1}\mu_\zeta(t)^{k-1}f_1(\mu_\zeta(t))}{(k-1)!\mu_\zeta(s)^{k-1}h(\mu_\zeta(t))}\right| dt, \qquad (2.10)$$

and, using

$$\begin{aligned} |(\mu_\zeta(s) - \mu_\zeta(t))^{k-1}| &\leq \sum_{p=0}^{k-1} \binom{k-1}{p} |\mu_\zeta(s)^p \mu_\zeta(t)^{k-1-p}| \\ &\leq k(k-1)!|\mu_\zeta(s)^{k-1}\mu_\zeta(t)^{k-1}|, \end{aligned}$$

we arrive at

$$|f_1(\mu_\zeta(s))| \leq W(s) = c + \int_0^s k|\mu_\zeta(t)^{2k-2}f_1(\mu_\zeta(t))h(\mu_\zeta(t))^{-1}|dt.$$

Hence,

$$\begin{aligned} \frac{dW}{ds} &\leq k|\mu_\zeta(s)|^{2k-2}|f_1(\mu_\zeta(s))||h(\mu_\zeta(s))|^{-1} \\ &\leq k|\mu_\zeta(s)|^{2k-2}|h(\mu_\zeta(s))|^{-1}W(s) \end{aligned}$$

and so

$$W(s) \leq c\exp\left(\int_0^s k|\mu_\zeta(t)|^{2k-2}|h(\mu_\zeta(t))|^{-1}dt\right) \leq c,$$

using (2.8), which gives $f_1(z) = O(1)$, so

$$f(z) = O(|z|^{k-1})$$

for $z \in C$.

2.3 Proof of Theorem 2.1.3

We assume the existence of a transcendental meromorphic function f of finite order as in the hypothesis, in particular satisfying (2.1) and such that, for some $k \in \mathbb{N}$, $f^{(k)}/f$ is transcendental. We set h, h_1 and f_1 as in (2.5) and (2.6), and note that $N_1 = (5 + d_1 + 4k)/2$. We begin by assuming both (2.2) and (2.3) are false. Then

$$T(r,f) - \frac{r^{1/2}}{2} \int_r^\infty \frac{N(t,f)}{t^{3/2}} dt - (k-1)\log r \to \infty \tag{2.11}$$

and

$$T(r, f^{(k)}/f) - \frac{r^{1/2}}{2} \int_r^\infty \frac{N(t, f/f^{(k)})}{t^{3/2}} dt - \frac{5 + d_1 + 4k}{2} \log r \to \infty \tag{2.12}$$

as $r \to \infty$, and straightforward integration by parts gives

$$m(r, f_1) - r^{1/2} \int_r^\infty \frac{n(t, f_1)}{t^{3/2}} dt \to \infty \tag{2.13}$$

and

$$m(r, h_1) - r^{1/2} \int_r^\infty \frac{n(t, h_1)}{t^{3/2}} dt \to \infty, \tag{2.14}$$

respectively. Hence, by Lemma 2.2.1, we have paths γ_{f_1} and γ_{h_1} tending to infinity on which f_1 and h_1 tend to infinity respectively.

Choose $r_0 \in [1, 2]$ such that h_1 is finite on $|z| = r_0$, and choose a large $R = T_1 > M(r_0, h_1)$ satisfying the conclusions of Lemma 2.2.2. By the existence of

γ_{h_1}, there exists an unbounded component, V, say, of the set $\{z : |h_1(z)| > T_1\}$, and V is contained in $\{z : |z| \geq r_0\}$ by the choice of T_1. We consider two cases:

Case (i) ∂V has no unbounded component. Then, by the choice of T_1, ∂V consists of countably many Jordan curves, and V is the only unbounded component of $\{z : |h_1(z)| > T_1\}$. By Lemma 2.2.5, $f_1(z) = O(1)$ as $z \to \infty$ in V, and this contradicts the existence of the path γ_{f_1}.

Case (ii) ∂V has an unbounded component. If this is so, we have a path tending to infinity on which $|h_1(z)| = T_1$, and this, together with (2.14), enables us to use Lemma 2.2.1 to obtain a function v_{h_1}, subharmonic, non-constant and non-negative in the plane, such that

$$v_{h_1}(z) \leq \log^+ |h_1(z)| + O(1) \tag{2.15}$$

for all z. By Lemma 2.2.3 and the result of Lewis, Rossi and Weitsman (Theorem 1.6.3), there exists a path Γ_1 tending to infinity such that

$$\frac{v_{h_1}(z)}{\log |z|} \to +\infty \tag{2.16}$$

as z tends to infinity on Γ_1, and

$$\int_{\Gamma_1} e^{-\delta v_{h_1}(z)} |dz| < \infty \tag{2.17}$$

for all $\delta > 0$. Equation (2.15) implies that

$$\frac{\log |h_1(z)|}{\log |z|} \to +\infty$$

on Γ_1 also. So there exists an unbounded component U_1 of $\{z : |h_1(z)| > T_1\}$ such that $\Gamma_1 \backslash U_1$ is bounded. On U_1, we have by Lemma 2.2.5 that f_1 is bounded by some finite positive constant, M, say, and v_{h_1} is unbounded since $\Gamma_1 \cap U_1$ is unbounded. However, $v_{h_1} \leq \log T_1 + O(1) \leq M_1$, say, on ∂U_1. In particular, $|f_1(z)| \leq M$ as $z \to \infty$ on Γ_1, and so there exists, by Lemma 2.2.1, a non-constant, non-negative subharmonic function v_{f_1} such that

$$v_{f_1}(z) \leq \log^+ |f_1(z)| + O(1).$$

Lewis, Rossi and Weitsman's result (Theorem 1.6.3) shows there is a path Γ_2 tending to infinity on which v_{f_1} tends to infinity. On U_1, clearly $v_{f_1} \leq M_2$, say, so we take an unbounded component U_2 of the set $\{z : v_{f_1} > M_2 + 1\}$ (such a component must exist by the existence of the path Γ_2) and we have immediately that U_1 and U_2 are disjoint. We therefore have the required conditions for the employment of Lemma 2.2.4. We obtain, for large r,

$$\log B(r, v_{h_1}) + \log B(r, v_{f_1}) \geq 2 \log r - O(1), \tag{2.18}$$

where $B(r, v) = \sup\{v(z) : |z| = r\}$. Using Poisson's inequality for subharmonic functions, with $R = 2r$, we have

$$\begin{aligned} B(r, v_{h_1}) &\leq \frac{3}{2\pi} \int_0^{2\pi} v_{h_1}(2re^{i\phi}) d\phi \\ &\leq 3m(2r, h) + O(1) \leq O(T(2r, f)), \end{aligned}$$

with a similar result holding for $B(r, v_{f_1})$. We therefore obtain from (2.18) that

$$2 \log T(2r, f) \geq 2 \log r - O(1),$$

so

$$\frac{T(2r, f)}{r} \geq c,$$

where the constant is positive. However, this contradicts (2.1), and so at least one of (2.2) or (2.3) holds.

Chapter 3

Differential polynomials

3.1 Introduction

In [22, p. 60], Hayman proves the following striking analogue of Picard's theorem.

Theorem 3.1.1 *Let f be transcendental and meromorphic in the plane, and let $p \geq 1$ be an integer. Then*

$$T(r,f) \leq \left(2+\frac{1}{p}\right) N\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{p}\right) \overline{N}\left(r,\frac{1}{f^{(p)}-1}\right) + S(r,f) \qquad (3.1)$$

as $r \to \infty$.

Theorem 3.1.1 implies "Hayman's alternative", which states that, for transcendental f, either every finite value is taken infinitely often, or $f^{(p)}$ takes every finite value, except possibly zero, infinitely often. Since Hayman's theorem requires only two values, and Picard's theorem requires three, this result is all the more remarkable.

Consider now a transcendental meromorphic function f and its first p derivatives. A *differential polynomial* P of f is defined by

$$P(z) = \sum_{k=1}^{n} \phi_k(z),$$

where

$$\phi_k(z) = \alpha_k(z) \prod_{j=0}^{p} \left(f^{(j)}(z)\right)^{S_{kj}},$$

$\alpha_k \not\equiv 0$, the S_{kj} are non-negative integers and $T(r, \alpha_k) = S(r, f)$ for all k. Let

$$\overline{d}(P) = \max_{1 \le k \le n} \left\{ \sum_{j=0}^{p} S_{kj} \right\} \quad \text{and} \quad \underline{d}(P) = \min_{1 \le k \le n} \left\{ \sum_{j=0}^{p} S_{kj} \right\}.$$

Then $\overline{d}(P)$ is the degree of P, while $\underline{d}(P)$ is the minimal degree of the constituent differential monomials. If $\underline{d}(P) = \overline{d}(P)$, P is said to be *homogeneous*, and *inhomogeneous* otherwise. Chuang [10] extends the idea of Theorem 3.1.1 to differential polynomials as follows.

Theorem 3.1.2 *Let f be transcendental and meromorphic, and let P be a non-constant differential polynomial in f such that $\underline{d}(P) > 1$. Then we have*

$$T(r, f) \le \frac{\underline{d}(P)}{\underline{d}(P) - 1} N\left(r, \frac{1}{f}\right) + \frac{1}{\underline{d}(P) - 1} \overline{N}\left(r, \frac{1}{P - 1}\right) + S(r, f). \qquad (3.2)$$

Interestingly, Chuang's proof is more straightforward than Hayman's. We show that in fact Theorem 3.1.2 can be improved, in the sense that the $N(r, 1/f)$ term in equation (3.2) can be replaced by $\overline{N}(r, 1/f)$ (which counts poles without regard to multiplicity), but with a different multiplicative constant.

Theorem 3.1.3 [1] *Let f be a transcendental meromorphic function, let P be a non-constant differential polynomial in f such that $\underline{d}(P) > 1$, and let*

$$Q = \max_{1 \le k \le n} \left\{ \sum_{j=1}^{p} j S_{kj} \right\}.$$

[1]A paper containing the proof of Theorem 3.1.3 has been accepted for publication and is to appear in the journal *Computational Methods and Function Theory.*

Then

$$T(r,f) \leq \frac{Q+1}{\underline{d}(P)-1}\overline{N}\left(r,\frac{1}{f}\right) + \frac{1}{\underline{d}(P)-1}\overline{N}\left(r,\frac{1}{P-1}\right) + S(r,f).$$

3.2 Proof of Theorem 3.1.3

The starting point of the proof of Theorem 3.1.3 is the penultimate statement in Chuang's proof of Theorem 3.1.2.

Lemma 3.2.1 *Let f be transcendental and meromorphic, and let P be a non-constant differential polynomial in f with $\underline{d}(P) \geq 1$. Then*

$$\underline{d}(P)T(r,f) \leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{P-1}\right)+\underline{d}(P)N\left(r,\frac{1}{f}\right)-N_0\left(r,\frac{1}{P'}\right)+S(r,f), \tag{3.3}$$

where $N_0(r,1/P')$ counts only those zeros of P' not corresponding to multiple zeros of $P-1$. When $\underline{d}(P) = \overline{d}(P) = 1$, Lemma 3.2.1 reduces to Theorem 3.2 of [22].

Proof The result is due to Chuang [10], but the proof given here is shorter. Suppose $|z| = r$, and $|f(z)| \leq 1$. Since

$$\sum_{j=0}^{p} S_{kj} \geq \underline{d}(P)$$

for every k, we have

$$\left|\frac{P(z)}{f(z)^{\underline{d}(P)}}\right| \leq \sum_{k=1}^{n} |\alpha_k(z)| \prod_{j=1}^{p} \left|\frac{f^{(j)}(z)}{f(z)}\right|^{S_{kj}}.$$

Hence, writing

$$\frac{1}{f(z)^{\underline{d}(P)}} = \frac{P(z)}{f(z)^{\underline{d}(P)}}\frac{1}{P(z)}$$

we obtain

$$\underline{d}(P)m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{1}{P}\right)+S(r,f), \tag{3.4}$$

since, as is well-known, the proximity function of any logarithmic derivative of f is always $S(r,f)$ (see section 1.2). Using (3.4) and Nevanlinna's first and second fundamental theorems (Theorems 1.2.2 and 1.2.5), we have

$$\begin{aligned}\underline{d}(P)m\left(r,\frac{1}{f}\right) &\le T(r,P)-N\left(r,\frac{1}{P}\right)+S(r,f)\\ &\le N(r,P)+N\left(r,\frac{1}{P-1}\right)-N_1(r,P)+S(r,f), \quad (3.5)\end{aligned}$$

in which

$$N_1(r,P)=N(r,P)-\overline{N}(r,P)+N\left(r,\frac{1}{P'}\right)\ge 0$$

is the branching term of Nevanlinna theory (see Theorem 1.2.5). Since

$$\overline{N}\left(r,\frac{1}{P-1}\right)-N_0\left(r,\frac{1}{P'}\right)=N\left(r,\frac{1}{P-1}\right)-N\left(r,\frac{1}{P'}\right),$$

equation (3.5) gives

$$\begin{aligned}\underline{d}(P)T(r,f) &= \underline{d}(P)T\left(r,\frac{1}{f}\right)+O(1)\\ &\le \underline{d}(P)N\left(r,\frac{1}{f}\right)+\overline{N}(r,P)+\overline{N}\left(r,\frac{1}{P-1}\right)-N_0\left(r,\frac{1}{P'}\right)\\ &\quad +S(r,f),\end{aligned}$$

from which (3.3) follows at once.

Note that Theorem 3.1.2 follows from Lemma 3.2.1 immediately, since

$$\overline{N}(r,f)\le T(r,f).$$

However, examination of the terms Chuang discards proves fruitful.

To prove Theorem 3.1.3, we estimate the term $\underline{d}(P)N(r,1/f)-N_0(r,1/P')$. First, we write

$$\frac{1}{f^{\underline{d}(P)}}=\frac{1}{P}\sum_{k=1}^{n}\alpha_k f^{v_k}\prod_{j=1}^{p}\left(\frac{f^{(j)}}{f}\right)^{S_{kj}}, \tag{3.6}$$

in which

$$v_k = \sum_{j=0}^{p} S_{kj} - \underline{d}(P) \geq 0.$$

Suppose that z_0 is a zero of f of multiplicity $m > 0$ and a zero of P of multiplicity μ (if P is non-zero at z_0, set $\mu = 0$), and that the α_k have poles of multiplicity $a_k \geq 0$ at z_0 (if $\alpha_k(z_0)$ is finite, set $a_k = 0$). The multiplicity of the pole of $f^{-\underline{d}(P)}$ at z_0 is clearly $\underline{d}(P)m$. It is well-known that the multiplicity of any pole of the function $f^{(k)}/f$, $k \in \mathbb{N}$, is at most k, so equation (3.6) implies that

$$\underline{d}(P)m \leq \mu + \max_{1 \leq k \leq n} \{a_k\} + \max_{1 \leq k \leq n} \left\{ \sum_{j=1}^{p} j S_{kj} \right\} \leq \mu + Q + \sum_{k=1}^{n} a_k, \tag{3.7}$$

Now let m_0 be the contribution of z_0 to $\underline{d}(P)n(r, 1/f) - n_0(r, 1/P')$, and consider two possible cases:

Case (i) $\mu = 0$. Here, equation (3.7) gives

$$m_0 \leq \underline{d}(P)m \leq Q + \sum_{k=1}^{n} a_k.$$

Case (ii) $\mu > 0$. In this case, $P(z_0) = 0$, and in particular, P is not equal to 1 at z_0, so z_0 contributes $\mu - 1 \geq 0$ to $n_0(r, 1/P')$, and equation (3.7) gives

$$m_0 \leq \underline{d}(P)m - (\mu - 1) \leq Q + \sum_{k=1}^{n} a_k + 1.$$

Hence, in either case,

$$\underline{d}(P)N\left(r, \frac{1}{f}\right) - N_0\left(r, \frac{1}{P'}\right) \leq (Q+1)\overline{N}\left(r, \frac{1}{f}\right) + \sum_{k=1}^{n} N(r, \alpha_k). \tag{3.8}$$

Substituting (3.8) into equation (3.3), and noting that

$$\overline{N}(r, f) \leq T(r, f) \quad \text{and} \quad N(r, \alpha_k) = S(r, f),$$

we obtain

$$T(r,f) \le \frac{Q+1}{\underline{d}(P)-1}\overline{N}\left(r,\frac{1}{f}\right) + \frac{1}{\underline{d}(P)-1}\overline{N}\left(r,\frac{1}{P-1}\right) + S(r,f)$$

and the proof is complete.

Chapter 4

The Bergweiler-Eremenko theorem for finite lower order

4.1 Introduction

Recall the definitions of direct and indirect transcendental singularities from section 1.9.4. The following is proved in [8]:

Theorem 4.1.1 (The Bergweiler-Eremenko theorem) *Let f be transcendental and meromorphic of finite order in the plane, such that the inverse function f^{-1} has an indirect singularity over a. Then a is a limit point of critical values of f.*

This result is in fact true for transcendental meromorphic f of finite *lower* order.

Theorem 4.1.2 *Let f be transcendental and meromorphic of finite lower order in the plane, such that the inverse function f^{-1} has an indirect singularity over a. Then a is a limit point of critical values of f.*

Following [8], it suffices to prove the following weaker result:

Theorem 4.1.3 [1] *Let f be transcendental and meromorphic of finite lower order in the plane, and suppose that the inverse function f^{-1} has an indirect transcendental singularity over 0. Then for every real positive t, the component $C(t)$ contains infinitely many zeros of f'.*

4.2 Results needed for Theorem 4.1.3

The first results in this section, relating to the obtaining of an infinite set of asymptotic values of f, are strongly reminiscent of [8]. We then proceed to find components on which f' is small, but because we have only assumed finite lower order, the subsequent analysis is of a local nature.

Lemma 4.2.1 *Suppose that f is transcendental and meromorphic in the plane and that the inverse function f^{-1} has an indirect transcendental singularity over 0 such that $C(\varepsilon)$, for some $\varepsilon > 0$, contains no zeros of f'. Let $z_1 \in C(\varepsilon)$, with $f(z_1) = 0$. Then there exists a with $0 < |a| = r < \varepsilon$ and a simply connected domain $D \subseteq C(\varepsilon)$ such that f maps D univalently onto $B(0, r)$, and D contains a path σ tending to infinity on which $f(z) \to a$ as $z \to \infty$, mapped by f onto the line segment $w = ta$, $0 \leq t < 1$.*

[1] The proof of this result appears in [27].

Proof Let g be that branch of the inverse function f^{-1} which maps 0 to z_1, and let r be the supremum of positive real t such that g extends to be analytic in $B(0,t)$. Clearly $r > 0$, since f is univalent on a neighbourhood of z_1, as there are no critical points in $C(\varepsilon)$. Further, g is analytic and univalent on $B(0,r)$, since if $g(w_1) = g(w_2)$, we have

$$w_1 = f(g(w_1)) = f(g(w_2)) = w_2.$$

Therefore, $D = g(B(0,r))$ is a simply connected domain and $|f(z)| \to r$ as $z \to \partial D$, and so D is that component of the set $\{z : |f(z)| < r\}$ which contains z_1. It follows that $r < \varepsilon$, since otherwise we would have $C(\varepsilon) \subseteq D$, which contradicts the fact that $C(\varepsilon)$ contains infinitely many zeros of f. In particular, we have $D \subseteq C(\varepsilon)$.

Now suppose that for every a with $|a| = r$, the branch g of f^{-1} can be analytically continued along the line segment $w = ta$, $0 \le t \le 1$. Then each such continuation defines an extension h_a of g to a disc $U_a = B(a, d_a)$, $d_a > 0$. If $U_a \cap U_b \neq \emptyset$, then $h_a = h_b = g$ on the non-empty intersection $U_a \cap U_b \cap B(0,r)$. Since $U_a \cap U_b$ is connected, we get $h_a = h_b$ on $U_a \cap U_b$. However, the circle $|w| = r$ is compact, and so can be covered by finitely many such U_a, from which it follows that g extends analytically to a disc $B(0, r_1)$, where $r_1 > r$, contradicting the definition of r.

It follows that there is some a with $|a| = r$ such that g does not admit analytic continuation along the path $w = ta$, $0 \le t \le 1$. Now, since $r < \varepsilon$, the path $g(ta)$, $0 \le t < 1$, lies in D, and hence its closure in the finite plane is contained in $C(\varepsilon)$. Hence, as $t \to 1-$, $g(ta)$ must tend either to infinity or to a critical point of f, the latter possibility being ruled out since f has no critical points in $C(\varepsilon)$, by assumption. So we may take the path $g(ta)$, $0 \le t < 1$, as our path σ, and the proof is complete.

Lemma 4.2.2 *Let f be as in Lemma 4.2.1, with $C(\varepsilon)$, for some $\varepsilon > 0$, containing no zeros of f'. Then we may find a sequence z_j tending to ∞, $z_j \in C(\varepsilon)$, and complex numbers a_j, $j \in \mathbb{N}$, with $|a_j| = r_j$ and $0 < r_{j+1} < \frac{1}{2}r_j$, $r_1 < \frac{1}{2}\varepsilon$, and pairwise disjoint simply connected domains $D_j \subseteq C(\varepsilon)$ such that f maps D_j univalently onto $B(0, r_j)$, with $f(z_j) = 0$ and such that each D_j contains a path $\sigma_j \to \infty$ on which $f(z) \to a_j$ as $z \to \infty$. The path σ_j is mapped by f onto the line segment $w = ta_j$, $0 \leq t < 1$.*

Proof We define a sequence of z_j, a_j inductively. Take $z_1 \in C(\frac{1}{2}\varepsilon)$ with $f(z_1) = 0$, and use Lemma 4.2.1 to obtain $a = a_1$, $D = D_1$ and $\sigma = \sigma_1$. Assuming that z_{n-1}, D_{n-1} have already been determined, we need only take $z_n \in C(\frac{1}{2}r_{n-1})$ with $f(z_n) = 0$ and $z_n \neq z_j$, $1 \leq j \leq n-1$, and determine D_n, r_n, a_n and σ_n by applying Lemma 4.2.1 again. In particular, we have $|a_n| = r_n < \frac{1}{2}r_{n-1}$, as required.

We assert that the D_j are pairwise disjoint. If $m < n$ and D_n meets D_m, then, since D_n is a component of the set $\{z : |f(z)| < r_n\}$, we have $D_n \subseteq D_m$, but this is a contradiction since $z_n \neq z_m$, and f is univalent on D_m.

Lemma 4.2.3 *Let $t > 0$, and let $\theta_j(t)$ equal the angular measure of the longest open arc of $S(0, t)$ which lies in D_j. As z tends to infinity on σ_j, we have*

$$\log \frac{r_j}{|f(z) - a_j|} \geq \int_{|z_j|}^{|z|} \frac{dt}{t\theta_j(t)} - \log 2.$$

Proof For a simply connected domain D with h mapping D univalently onto the unit disc Δ, the hyperbolic distance between two points in D, w_1 and w_2, is given by $[w_1, w_2]_D = [h(w_1), h(w_2)]_\Delta$ (see section 1.5). Now, in the case of D_j, we choose $f(z)/r_j$ as our function h, and taking the two points z_j and z, for some z on σ_j, we get

$$[z_j, z]_{D_j} = [0, h(z)]_\Delta = \log\left(\frac{1+|h(z)|}{1-|h(z)|}\right) \geq \int_{|z_j|}^{|z|} \frac{dt}{t\theta_j(t)}, \tag{4.1}$$

using Theorem 1.5.1. However, f maps σ onto the line segment $w = ta_j$, $0 \leq t < 1$, and so $1 - |h(z)| = |f(z) - a_j|/r_j$. Also, $\log(1 + |h(z)|) \leq 2$, and so the result follows from (4.1).

Lemma 4.2.4 *Let u lie on σ_j. Then there exists v on σ_j with $|u| \leq |v| \leq |u| + 1$, such that $|f(v) - a_j| \leq |f(u) - a_j|$ and $|f'(v)| \leq |f(u) - a_j|$.*

Proof We start at u, and follow the path σ_j in the direction in which $|f(z) - a_j|$ decreases, until the first point w with $|w| = |u| + 1$ is reached, denoting by γ that part of the curve σ_j between u and w. Then the inverse function $g = f^{-1}$ maps a sub-segment I of $[f(u), a_j)$ onto γ, and so

$$1 \leq |w - u| \leq \left|\int_I g'(\zeta) d\zeta\right| \leq |f(u) - a_j| \max\{|g'(\zeta)| : \zeta \in I\}.$$

Since $|g'| = 1/|f'|$ here, we have our v.

Lemma 4.2.5 *Suppose there exists a sequence $r_n \to \infty$ with $T(r_n^7, f) \leq r_n^M$ for some $M > 0$. Then*

$$T(r_n^6, f') \leq 3r_n^M \quad \textit{and} \quad n(r_n^5, f') + n(r_n^5, 1/f') \leq 6r_n^M$$

for large enough r_n.

Proof We have

$$\begin{aligned} T(r_n^6, f') &\leq T(s_n, f') \\ &\leq 2N(s_n, f) + m(s_n, f') \\ &\leq 2T(s_n, f) + m\left(s_n, \frac{f'}{f}\right) \\ &\leq 2T(s_n, f) + S(s_n, f) \end{aligned}$$

where $r_n^6 \le s_n \le r_n^7$, and s_n is chosen so that $S(s_n, f) = o(T(s_n, f))$. Then

$$T(r_n^6, f') \le 3T(r_n^7, f) \le 3r_n^M$$

if r_n is large enough.

Now,

$$\begin{aligned} N(r_n^6, f') &= \int_0^{r_n^6} (n(t, f') - n(0, f'))\frac{dt}{t} + n(0, f')\log r_n^6 \\ &\ge \int_{r_n^5}^{r_n^6} (n(t, f') - n(0, f'))\frac{dt}{t} + n(0, f')\log r_n^6 \\ &\ge (n(r_n^5, f') - n(0, f'))\log r_n + n(0, f')\log r_n^6 \\ &\ge n(r_n^5, f')\log r_n \end{aligned}$$

for $r_n \ge 1$. Since $T(r, f') = T(r, 1/f') + O(1)$, we obtain

$$n(r_n^5, f')\log r_n \le N(r_n^6, f') \le T(r_n^6, f') \le 3r_n^M$$

and

$$n(r_n^5, 1/f')\log r_n \le N(r_n^6, 1/f') \le T(r_n^6, f') + O(1) \le 4r_n^M$$

for large r_n, so

$$\begin{aligned} 2(n(r_n^5, f') + n(r_n^5, 1/f')) &\le (n(r_n^5, f') + n(r_n^5, 1/f'))\log r_n \\ &\le 12r_n^M. \end{aligned}$$

Lemma 4.2.6 *Let f and the sequence (r_n) be as in Lemma 4.2.5. Denote the zeros and poles of f' in the region $r_n^{1/4} \le |z| \le r_n^4$ by w_j, $j = 1, \ldots, N$, and surround each w_j by a disc of radius r_n^{-M-1}. The sum of the radii of these discs, and hence the area of their union U_n, tends to 0 as $n \to \infty$. Further, for $z \notin U_n$ and $r_n^{1/3} \le |z| \le r_n^3$,*

$$\left|\log\left|\frac{1}{f'(z)}\right|\right| \le r_n^{M+3}$$

provided r_n is large enough.

Proof The result follows by elementary means from the Poisson-Jensen formula [22] and Lemma 4.2.5.

Lemma 4.2.7 *Let $0 < \varepsilon < \frac{1}{2}$ for the remainder of this section, and fix an integer $N > 3(M+5)$ so large that the function $G(z) = z^N f'(z)$ has $G(0) = 0$. Denote by $L(r, R, G)$ the total length of the level curves $|G(z)| = R$ in the region $|z| < r$. If n is large enough, there exists an $R_n \in (\varepsilon, 2\varepsilon)$ such that*

$$L(r_n^3, R_n, z^N f'(z)) \leq r_n^{M+4},$$

with no critical values of G on $|w| = R_n$.

Proof We start from the length-area principle (Theorem 1.9.1),

$$\int_0^\infty \frac{L(r_n^3, R, G)^2}{p(r_n^3, R, G)} \frac{dR}{R} \leq 2\pi^2 r_n^6, \tag{4.2}$$

where

$$p(r_n^3, R, G) = \frac{1}{2\pi} \int_0^{2\pi} n(r_n^3, Re^{i\phi}, G) d\phi$$

and $n(s, w, G)$ equals the number of roots in $B(0, s)$ of the equation $G(z) = w$. Now, for $\varepsilon < R < 2\varepsilon$,

$$\begin{aligned} n(r_n^3, Re^{i\phi}, G) &\leq N\left(r_n^4, \frac{1}{G(z) - Re^{i\phi}}\right) \\ &\leq T(r_n^4, G(z) - Re^{i\phi}) + \log 1/R \\ &\leq 3r_n^{M+1} \end{aligned}$$

for sufficiently large r_n, by Lemma 4.2.5. Hence

$$p(r_n^3, R, G) \leq 3r_n^{M+1}.$$

Using (4.2), we have

$$\int_\varepsilon^{2\varepsilon} \frac{L(r_n, R, G)^2}{p(r_n^3, R, G)} \frac{dR}{R} \leq 2\pi^2 r_n^6$$

and so there exist uncountably many $R_n \in (\varepsilon, 2\varepsilon)$ such that

$$L(r_n^3, R_n, z^N f'(z))^2 \leq \frac{2\pi^2 r_n^6 p(r_n^3, R_n, z^N f'(z))}{\log 2}.$$

We can take R_n such that there are no critical values on $|w| = R_n$. So

$$L(r_n^3, R_n, z^N f'(z))^2 \leq \frac{6\pi^2 r_n^{M+1} r_n^6}{\log 2} \leq r_n^{M+8},$$

provided r_n is large enough, and the result follows.

Lemma 4.2.8 *Let C be a component of the set*

$$\{z : r_n^{1/2} - 1 < |z| < r_n^2 + 1, |z^N f'(z)| < R_n\},$$

where N and R_n are as in Lemma 4.2.7. Then

$$diam\ f(C) = o(1)$$

as $n \to \infty$.

Proof Choose z_1, z_2 in C with $|z_1| \leq |z_2|$ and join them with a path σ consisting of part of the ray $\arg z = \arg z_1$, $|z| \geq |z_1|$, and part of the circle $|z| = |z_2|$. Form a path γ by replacing any part of σ which leaves $C \cup \partial C$ with an arc of ∂C. Denote by γ_1 those parts of γ which are part of the ray, by γ_2 the parts of γ which are part of the circle of radius $|z_2|$, by γ_3 the parts of γ which are parts of the circles of radius $r_n^{1/2} - 1$ or $r_n^2 + 1$, and finally by γ_4 those parts of γ which remain. Now,

$$\int_{\gamma_1} |f'(z)||dz| \leq \int_{r_n^{1/2}-1}^{\infty} \frac{R_n}{t^N} dt = o(1), \quad \int_{\gamma_2} |f'(z)||dz| \leq 2\pi |z_2| \frac{R_n}{|z_2|^N} = o(1)$$

and

$$\int_{\gamma_3} |f'(z)||dz| \leq 4\pi r_n^{-N/3+3} = o(1)$$

as $n \to \infty$. Since on γ_4 we have $|z^N f'(z)| = R_n$, we can use Lemma 4.2.7 to obtain

$$\int_{\gamma_4} |f'(z)||dz| \le L(r_n^3, R_n, z^N f'(z)) \sup_{z\in\gamma_4} |f'(z)| < r_n^{M+4-N/3} R_n = o(1)$$

by the choice of N, and since $R_n < 1$. Hence

$$\int_\gamma |f'(z)||dz| = o(1), \quad |f(z_1) - f(z_2)| = o(1)$$

as $n \to \infty$, and since z_1 and z_2 are arbitrary, we get diam $f(C) = o(1)$ as $n \to \infty$.

Lemma 4.2.9 *Let D be a domain not containing the origin nor any whole circle $S(0,t)$, $t > 0$, and let z_0 lie in D. Let $r \neq |z_0|$, let $\theta_D(t)$ denote the angular measure of $D \cap S(0,t)$, and let D_r be the component of $D \backslash S(0,r)$ which contains z_0. Then*

$$\omega(z_0, S(0,r), D_r) \le 6e^{1/2} \exp\left(-\pi \int_I \frac{dt}{t\theta_D(t)}\right),$$

where the range of integration I is given by

$$I = \begin{cases} [2|z_0|, r/2] & \text{if } r > 4|z_0| \\ [2r, |z_0|/2] & \text{if } 4r < |z_0|. \end{cases}$$

Proof The result for $r > 4|z_0|$ follows from Tsuji's estimate (Theorem 1.4.3), and the case $4r < |z_0|$ from the conformal invariance of harmonic measure (Theorem 1.4.2).

Lemma 4.2.10 *We fix a positive K, large compared to M and N, and take M_1 with $M_1 > K^2$. Suppose we have M_1 pairwise disjoint domains E_j, $j = 1, \ldots, M_1$. If $\theta_j(t)$ denotes the angular measure of $E_j \cap S(0,t)$, then at least $M_1/2$ of the E_j are such that*

$$\int_I \frac{dt}{t\theta_j(t)} > K \log r_n, \tag{4.3}$$

for both of the intervals $I = [4r_n, r_n^2/4], [4r_n^{1/2}, r_n/4]$.

Proof First, consider the case $I = [4r_n, r_n^2/4]$ and suppose (4.3) does not hold for M_2 of the sets E_j. Without loss of generality, suppose that these are $E_1, \ldots, E_{M_2}$. Writing $1 = \theta_j(t)^{1/2}\theta_j(t)^{-1/2}$, a standard application of the Cauchy-Schwarz inequality gives

$$M_2^2 \leq 2\pi \sum_{j=1}^{M_2} \theta_j(t)^{-1}, \quad \int_I M_2^2 \frac{dt}{t} \leq 2\pi \sum_{j=1}^{M_2} \int_I \frac{dt}{t\theta_j(t)},$$

since the E_j are pairwise disjoint, by assumption. Hence

$$M_2^2 \log(r_n/16) \leq 2\pi K M_2 \log r_n,$$

so

$$M_2 \leq 2\pi K + o(1)$$

as $n \to \infty$, and provided K and n are chosen large enough, we have $M_2 < K^2/4 < M_1/4$. The case $I = [4r_n^{1/2}, r_n/4]$ is similar, and so at least $M_1/2$ of the E_j satisfy inequality (4.3).

Lemma 4.2.11 *Suppose that* $1 \leq j \leq M_1$*, that* n *is large, and that* j *is such that the inequality (4.3) of Lemma 4.2.10 holds for* $\theta_j(t) = \theta_{D_j}(t)$*. Then there exists* $v_j = v_j(n)$ *on* σ_j *with* $r_n \leq |v_j| \leq r_n + 1$ *and*

$$|v_j|^N |f'(v_j)| = o(1)$$

as $n \to \infty$.

Proof By the choice of j,

$$\int_{r_n^{1/2}}^{r_n} \frac{dt}{t\theta_j(t)} > K \log r_n.$$

Pick r_n so large that $|z_j| < r_n^{1/2}$, pick $\hat{z}_n$ on σ_j with $|\hat{z}_n| = r_n$ and employ Lemma 4.2.3 to get

$$\begin{aligned}\log \frac{|a_j|}{|f(\hat{z}_n) - a_j|} &\geq \int_{|z_j|}^{|\hat{z}_n|} \frac{dt}{t\theta_j(t)} - \log 2 \\ &\geq \int_{r_n^{1/2}}^{r_n} \frac{dt}{t\theta_j(t)} - \log 2 \\ &> K \log(r_n/2).\end{aligned}$$

So

$$\log |a_j| - \log |f(\hat{z}_n) - a_j| > K \log(r_n/2)$$

and hence

$$\log |f(\hat{z}_n) - a_j| < \log |a_j| - K \log(r_n/2) < -K \log(r_n/2),$$

since $|a_j| < 1$. So

$$|f(\hat{z}_n) - a_j| < \frac{4^K}{r_n^K} = C r_n^{-K}, \quad \text{and} \quad r_n^N |f(\hat{z}_n) - a_j| = C r_n^{N-K} = o(1)$$

as $n \to \infty$. By Lemma 4.2.4, there exists a v_j on σ_j with $r_n \leq |v_j| \leq r_n + 1$ such that

$$|f'(v_j)| \leq |f(\hat{z}_n) - a_j| = o(r_n^{-N}) = o(|v_j|^{-N})$$

as $n \to \infty$.

4.3 Proof of Theorem 4.1.3

Assume the result is false, so that there is some $0 < \varepsilon < \frac{1}{2}$ such that $C(\varepsilon)$ contains no zeros of f'. Let D_j, a_j, and σ_j be as in Lemma 4.2.2. Since f has finite lower order, there exists a sequence $r_n \to \infty$ such that $T(r_n^7, f) \leq r_n^M$, for some finite M. Pick s_n and t_n such that $|s_n - r_n^{1/2}| \leq 1$, $|t_n - r_n^2| \leq 1$ and the conclusion of Lemma 4.2.6 holds on $|z| = s_n, t_n$ (we can certainly do this for large enough r_n, since if not we contradict the fact that the sum of radii of the circles in U_n is $o(1)$). Henceforth, K and N are as chosen in Lemmas 4.2.7 and 4.2.10, and $M_1 > K^2$. Set

$$h = \min\{|a_m|, |a_j - a_k| : 1 \leq m \leq 2M_1, 1 \leq j < k \leq 2M_1\},$$

and take $D_1, \ldots, D_{2M_1}$. By Lemma 4.2.10, at least M_1 of the D_j are such that

$$\int_I \frac{dt}{t\theta_{D_j}(t)} > K \log r_n,$$

for both of the intervals I specified in Lemma 4.2.10. Re-labelling if necessary, we assume that this holds for $j = 1, \ldots, M_1$. Using Lemma 4.2.11, we may therefore find, for large enough n, a v_j on each σ_j with $r_n \leq |v_j| \leq r_n + 1$ and $|v_j|^N |f'(v_j)| < \min\{R_n/2, h/4\}$, where $R_n \in (\varepsilon, 2\varepsilon)$ is as in Lemma 4.2.7. Each v_j is contained in a component C_j of the set

$$\{z : s_n < |z| < t_n, |z^N f'(z)| < R_n\}.$$

By Lemma 4.2.8, we can assume r_n to be so large that diam $f(C_j) < h/4$, $j = 1, \ldots, M_1$. Now, on a given C_j, we have

$$|f(z) - a_j| = |f(z) - f(v_j) + f(v_j) - a_j| < \text{diam } f(C_j) + h/4 < h/2,$$

and hence the C_j are pairwise disjoint. Now, using Lemma 4.2.10, we can pick a j, $1 \leq j \leq M_1$, such that equation (4.3) holds, where $\theta_j(t)$ is now equal to the angular measure of $S(0,t) \cap C_j$. On C_j we have

$$|f(z)| \leq |f(z) - a_j| + |a_j| \leq h/2 + |a_j| \leq \frac{3|a_j|}{2} < \varepsilon,$$

so C_j is contained in a component of $\{z : |f(z)| < \varepsilon\}$. Since v_j is on $\sigma_j \subseteq D_j$, C_j is contained in $C(\varepsilon)$. Hence there are no zeros of f' on C_j, and so the function

$$u(z) = \log \left| \frac{1}{z^N f'(z)} \right|$$

is subharmonic in C_j. Now, denoting the parts of ∂C_j which are arcs of the circle $|z| = s_n$ by A and parts of ∂C_j which are arcs of the circle $|z| = t_n$ by B, we have, using Lemma 4.2.6,

$$\limsup_{z \to x, x \in A \cup B} u(z) \leq r_n^{M+3}$$

and

$$\limsup_{z \to x, x \in \partial C_j \setminus (A \cup B)} u(z) = \log \frac{1}{R_n}.$$

Hence, by the two constants theorem (Theorem 1.4.1), we have, for z in C_j,

$$u(z) \leq (\omega(z, A, C_j) + \omega(z, B, C_j)) r_n^{M+3} + \omega(z, \partial C_j \setminus (A \cup B), C_j) \log \frac{1}{R_n}. \quad (4.4)$$

We note that C_j contains no circles $|z| = t$, since otherwise, because r_n is large, C_j meets one of the paths σ_k, $k \neq j$, at a point w such that $|f(w) - a_k| < h/2$. This implies that

$$|a_j - a_k| = |a_j - f(w) + f(w) - a_k| \leq |f(w) - a_j| + |f(w) - a_k| < h,$$

which contradicts the choice of h. We can therefore use Lemma 4.2.9 to estimate $\omega(z, A, C_j)$ and $\omega(z, B, C_j)$ as follows:

$$\begin{aligned} \omega(z, A, C_j) &\leq (6e^{1/2}) \exp\left(-\pi \int_{2s_n}^{|z|/2} \frac{dt}{t\theta_j(t)}\right) \\ \omega(z, B, C_j) &\leq (6e^{1/2}) \exp\left(-\pi \int_{2|z|}^{t_n/2} \frac{dt}{t\theta_j(t)}\right), \end{aligned}$$

provided $4s_n < |z| < t_n/4$. By equation (4.3) and the choice of j,

$$\exp\left(-\pi \int_{2s_n}^{|v_j|/2} \frac{dt}{t\theta_j(t)}\right) < \exp(-K\pi \log r_n) = r_n^{-\pi K},$$

with a similar result holding for the integral over $[2|v_j|, t_n/2]$. Substituting these results into equation (4.4), we obtain

$$u(v_j) \leq 12e^{1/2} r_n^{M+3-\pi K} + \log \frac{1}{R_n}. \quad (4.5)$$

However, we chose v_j so that $u(v_j) > \log 2/R_n$, and if we take large enough r_n, the right-hand side of (4.5) is less than $\log 3/2R_n$, say. This is a contradiction, and so the theorem is proved.

4.4 Proof of Theorem 4.1.2

Following Bergweiler and Eremenko, we show how Theorem 4.1.2 follows from Theorem 4.1.3. First, we need a lemma of Nevanlinna.

Lemma 4.4.1 (Nevanlinna, [32]) *Let f be transcendental and meromorphic in the plane, let $0 < S < \infty$ and let C be a component of the set $\{z : S < |f(z)| \leq \infty\}$. Let z_0 be a point in C such that $w_0 = f(z_0)$ is finite and $f'(z_0) \neq 0$, and let g be that branch of the inverse function f^{-1} which maps w_0 to z_0. Suppose g admits unrestricted analytic continuation in the annulus $S < |w| < \infty$, starting at w_0. Then C is simply connected and contains either exactly one pole of f (possibly multiple), or no poles of f but instead a path σ tending to infinity on which f tends to infinity.*

To prove Theorem 4.1.2, we first assume that $a = 0$, and that the theorem is not true. We may therefore find some $\varepsilon > 0$ such that the only critical points of f in $C(\varepsilon)$ are zeros of f. Since f has finite lower order, the Denjoy-Carleman-Ahlfors theorem (Theorem 1.9.3) tells us that f can have only finitely many direct transcendental singularities, and we may assume that ε is so small that there is no w in the annulus $0 < |w| < \varepsilon$ such that f^{-1} has a direct singularity over w.

Take z_0 in $C(\varepsilon)$ with $f(z_0) = w_0 \neq 0$, and let g be that branch of the inverse function f^{-1} which maps w_0 to z_0. We claim that g admits unrestricted analytic continuation in $0 < |w| < \varepsilon$, noting that if this is the case then Lemma 4.4.1 implies that $C(\varepsilon)$ is simply connected and contains at most one zero of f. This is a contradiction, since the singularity over 0 is indirect.

To prove the claim, take a path $\gamma : [0,1] \to \{w : \delta \leq |w| \leq \varepsilon - \delta\}$ with $\gamma(0) = w_0$ and $\delta > 0$ chosen to be small compared to $|w_0|$. Suppose that analytic continuation of g along γ is not possible; then there exists $S \in [0,1]$ such that as $t \to S-$, $z = g(\gamma(t))$ either tends to infinity or to a critical point z_1 of f with $\delta \leq |z_1| \leq \varepsilon - \delta$. However, the latter possibility is ruled out by the choice of ε. It follows that the path σ given by $0 \leq t \leq S$ is a path tending to infinity and lying in $C(\varepsilon)$, and on which $f(z) \to w_1$ as $z \to \infty$, with $\delta \leq |w_1| \leq \varepsilon - \delta$.

However, in this case an unbounded subpath of σ lies in a component C' of the set $\{z : |f(z) - w_1| < \delta/2\}$, and $C' \subseteq C(\varepsilon)$, and hence f' has no zeros on C'. Further, the singularity over w_1 must be indirect, since we have excluded direct singularities with $0 < |w| < \varepsilon$, and so we have a contradiction of Theorem 4.1.3. Thus, since δ is arbitrarily small, the claim is proved.

4.5 Corollaries

The following results are applications of Theorem 4.1.2, and are previously existing results stated with relaxed conditions. Corollary 4.5.1 was proved originally in [14], and Corollaries 4.5.2 and 4.5.3 in [11]; to see how Theorem 4.1.2 improves them, the reader is referred to the alternative proofs given in [8]. Corollary 4.5.4 is a generalisation of a result from [16], whose proof follows that of the original exactly, save for the employment of Theorem 4.1.2 in place of Theorem 4.1.1.

Corollary 4.5.1 *Let f be transcendental and meromorphic with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

Then f' has infinitely many zeros.

Corollary 4.5.2 *Let f be transcendental and entire, with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

Then f'/f has infinitely many zeros.

Corollary 4.5.3 *Let f be transcendental and meromorphic with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{\sqrt{r}} = 0.$$

Then f'/f has infinitely many zeros.

Corollary 4.5.4 *Let f be transcendental and meromorphic in the plane, with finite lower order. If f has only zeros of order at least $k \geq 2$, then $f^{(j)}$ assumes every finite non-zero value infinitely often for $j = 1, \ldots, k-1$.*

Chapter 5

Unbounded analytic functions on plane domains

5.1 Introduction

The well-known result of Iversen (Theorem 1.6.2) states that if f is a non-constant entire function, there exists a path γ which tends to infinity in $\mathbb{C}$ along which f tends to infinity. Since a transcendental entire function f has a non-constant entire first derivative f', Rubel considers in [36] the question of whether, given a transcendental entire function f, there exists a path γ tending to infinity in $\mathbb{C}$ along which both f and f' tend to infinity. This question was answered positively for functions of finite order by Langley in [29], although the infinite order case is still unsolved.

Rubel's paper raises a number of interesting side-issues and concludes with a list of open questions. In trying to solve the original problem he proves that there exists, for transcendental entire f, a sequence (z_m) in $\mathbb{C}$ such that $f^{(j)}(z_m) \to \infty$ as $m \to \infty$, for all $j \geq 0$. He also shows, using Nevanlinna theory, that an analogous result holds on the unit disc Δ, i.e. there exists a sequence $(z_m) \subseteq \Delta$ on which f and all its derivatives tend to infinity, for analytic functions f which grow sufficiently rapidly on the disc. Whether such a sequence exists for general unbounded f on Δ was left as an open question by Rubel, answered positively by Gordon in [18]. Gordon's result leads naturally on to considering whether similar sequences exist on more general plane domains.

We say a function f on a domain D is *strongly unbounded* on D if there exists a sequence (z_m) in D on which f and all its derivatives tend to infinity. If all unbounded analytic functions on D are strongly unbounded there, we say D is a *Rubel domain.* Clearly, the function $f(z) = z$ shows that a Rubel domain must be bounded, although an example has been produced of a bounded domain D and an analytic function f unbounded on D but with bounded first derivative (see [36]).

Let E be a subset of a domain D. Define $R(E, z)$ as the set of straight line segments contained entirely in D which join the point z to points in E. We define a *δ-visible* set as follows: for a positive constant δ, a subset E of $D \subseteq \mathbb{C}$ is δ-visible in D if the angular spread of $R(E, z)$ is at least δ for all $z \in D \backslash E$. Gordon's approach can be generalised easily to give the following result.

Theorem 5.1.1 *Let D be a bounded domain in $\mathbb{C}$. If D contains a δ-visible compact set, then D is a Rubel domain.*

If we have a domain D with a sufficiently well-behaved conformal map, then a further generalisation of Gordon's method can be persuaded to give results.

In this direction we obtain the following.

Theorem 5.1.2 [1] *Let D be a bounded domain in $\mathbb{C}$. If D is a quasidisk, then D is a Rubel domain.*

A detailed discussion of the geometry of quasidisks can be found in [33]: in particular, a bounded domain D is a quasidisk if and only if the conformal map $f : \Delta \to D$ extends to a quasiconformal mapping in the extended plane, fixing infinity.

As we mentioned in section 1.8.1, one characteristic of those Jordan domains which are quasidisks is that their boundaries must contain no cusps [33, p.107]. However, it is not difficult to construct a domain D which contains a δ-visible compact set and is such that ∂D contains an inward-pointing cusp. By Theorem 5.1.1, such a domain is a Rubel domain, and hence Theorem 5.1.2 gives a sufficient, but not necessary, condition for D to be a Rubel domain.

It would be interesting to know whether a bounded starlike domain is necessarily a Rubel domain. As is mentioned in section 1.8.2, a quasiconformal extension to the conformal map associated with an arbitrary *strongly* starlike domain is explicitly given in [15], and hence by Theorem 5.1.2 strongly starlike domains are Rubel domains, but the general starlike case remains unsolved.

[1] A paper containing proofs of Theorems 5.1.1 and 5.1.2 has been accepted for publication and is to appear in the journal *Mathematatika*.

5.2 A sufficient condition for strong unboundedness

Lemma 5.2.1 *Let D be a domain contained in the disc $B(0, M)$, $M > 1$, and let f be analytic and unbounded on D.*

(I) Suppose $k \geq 0$ and that we are given a sequence (z_m) in D such that

$$f^{(j)}(z_m) \to \infty \text{ as } m \to \infty, \quad \forall j \in \{0, \ldots, k\}. \tag{5.1}$$

For each m, expand f about the point z_m with Taylor's theorem to obtain

$$f(z) = T_m(z) + R_m(z), \tag{5.2}$$

where

$$T_m(z) = f(z_m) + f'(z_m)(z - z_m) + \frac{f''(z_m)}{2!}(z - z_m)^2 + \ldots + \frac{f^{(k)}(z_m)}{k!}(z - z_m)^k \tag{5.3}$$

and

$$R_m(z) = \frac{1}{k!}\int_{z_m}^{z} f^{(k+1)}(\zeta)(z - \zeta)^k d\zeta, \tag{5.4}$$

and suppose the following holds:

(a) there exists a real sequence (β_m) satisfying

$$\beta_m \to \infty \text{ as } m \to \infty, \quad \beta_m < |f^{(j)}(z_m)|, \quad \forall j \in \{0, \ldots, k\}. \tag{5.5}$$

(b) there exists a compact set $E \subset D$, and

(c) there exist paths $\gamma_m \subset D$ joining z_m to E such that

(i) $|T_m^{(j)}(z)| > \beta_m$ for $z \in \gamma_m$, $j \in \{0, \ldots, k\}$, and

(ii) there exists a constant C, independent of m and assumed greater than 1, such that the length of γ_m, denoted $L(\gamma_m)$, is less than C.

Then we can find, for each large m, a point z'_m in D such that $|f^{(j)}(z'_m)| > \beta_m/2$ for $j \in \{0, \ldots, k\}$ and $|f^{(k+1)}(z'_m)| \geq \beta_m/2C(2M)^k$.

(II) If, for every $k \geq 0$ and every sequence (z_m) in D satisfying (5.1), there exist (β_m), E and γ_m as in (I)(a), (b) and (c), then f is strongly unbounded on D.

Proof The proof is adapted from [18]. If $|f^{(k+1)}(z_m)| \geq \beta_m/2C(2M)^k$, taking z_m as the point z'_m gives the conclusion immediately, so we assume

$$|f^{(k+1)}(z_m)| < \beta_m/2C(2M)^k.$$

Denote by $\tilde{z}_m$ the first intersection of the path γ_m with E. Then, for large m, since $|f^{(k)}(z)|$ is bounded on E,

$$\begin{aligned}
\max_{z\in\gamma_m} |f^{(k+1)}(z)| &\geq \frac{1}{L(\gamma_m)} \left| \int_{\gamma_m} f^{(k+1)}(\zeta) d\zeta \right| \\
&\geq \frac{1}{C} \left| f^{(k)}(z_m) - O(1) \right| \\
&\geq \frac{\beta_m}{2C(2M)^k}.
\end{aligned}$$

We follow the path γ_m from z_m, denoting by z'_m the first point we reach at which $|f^{(k+1)}(z)| = \beta_m/2C(2M)^k$. Then, for $j \in \{0, 1, \ldots, k\}$,

$$\begin{aligned}
|R_m^{(j)}(z'_m)| &= \left| \frac{1}{(k-j)!} \int_{z_m}^{z'_m} f^{(k+1)}(\zeta)(z-\zeta)^{k-j} d\zeta \right| \\
&\leq \frac{1}{(k-j)!} \int_{z_m}^{z'_m} \frac{\beta_m}{2C(2M)^k} (2M)^{k-j} |d\zeta| \\
&\leq \frac{\beta_m}{2},
\end{aligned}$$

by property (I)(c)(ii), taking the integration over the subpath of γ_m which joins z_m to z'_m. So, by property (I)(c)(i),

$$|f^{(j)}(z'_m)| \geq |T_m^{(j)}(z'_m)| - |R_m^{(j)}(z'_m)| > \beta_m/2$$

for $j \in \{0, 1, \ldots, k\}$, and the proof is complete.

5.3 Proof of Theorem 5.1.1

It suffices to show that, for every $k \geq 0$ and every sequence (z_m) as in (5.1), there exist a sequence β_m as in (5.5), a compact set $E \subset D$ and a set of paths γ_m satisfying conditions (I)(c)(i) and (ii) of Lemma 5.2.1. Denote the compact δ-visible set in D by E. We will take as the paths γ_m one of the straight line segments joining z_m to E; since D is a bounded domain, such a line segment cannot have length greater than $2M$, and we immediately have condition (I)(c)(ii).

Gordon proves a lemma in [18] which states that, given a polynomial Q of degree l, a point ζ such that $Q(\zeta) \neq 0$ and a positive β with $\beta < |Q(\zeta)|$, the angular size of the set $\{z \in \mathbb{C} : |Q(z)| \leq \beta\}$, as measured from ζ, is bounded above by

$$2\pi l \left(\frac{\beta}{|Q(\zeta)|}\right)^{1/l}.$$

Given $k \geq 0$, suppose there exists a sequence (z_m) in D on which $f^{(j)}$ tends to infinity, for $j \in \{0, \ldots, k\}$. Choose any real sequence β_m such that (5.5) holds and $\beta_m = o(|f^{(j)}(z_m)|)$ as $m \to \infty$ for $j \in \{0, \ldots, k\}$. For $j \in \{0, \ldots, k\}$, note that $T_m^{(j)}(z_m) = f^{(j)}(z_m)$ and define

$$D_j = \{z \in \mathbb{C} : |T_m^{(j)}(z)| \leq \beta_m\}.$$

Then the angular size, as measured from z_m, of any one of the D_j is not more than

$$2\pi l \left(\frac{\beta_m}{|f^{(j)}(z_m)|} \right)^{1/l}, \quad l = \deg(T_m^{(j)}) \leq k,$$

which tends to zero as m tends to infinity. For m so large that this angular size is less than $\delta/(k+1)$, there is guaranteed to be a ray joining z_m to E which does not intersect with any of the sets D_j. On such a path, $|T_m^{(j)}(z)| > \beta_m$ for $j \in \{0, \ldots, k\}$, so condition (I)(c)(i) of Lemma 5.2.1 is satisfied and we have our paths γ_m.

5.4 Proof of Theorem 5.1.2

Let D be a quasidisk contained in the disc $B(0, M)$. By [33, p.94], there exists a K-quasiconformal ϕ mapping $\mathbb{C}$ onto $\mathbb{C}$ with $\phi(\Delta) = D$ and ϕ conformal on Δ. Let $E \subset D$ be the image under ϕ of the closed disc $\overline{B(0, \frac{1}{2})}$.

Lemma 5.4.1 *Let Q be a polynomial of degree at most l, and let $\zeta \in D$ with $Q(\zeta) \neq 0$. Suppose we are given $l+1$ distinct points $w_0, \ldots, w_l$, $w_j \neq \zeta$, such that $|Q(w_j)| \leq |Q(\zeta)|^{1/2}$ and the angle subtended at $\eta = \phi^{-1}(\zeta)$ by any pair u_i, $u_{i'}$ $(i \neq i')$ of the points $u_j = \phi^{-1}(w_j)$ is at least α, where $0 < \alpha < \pi/4$. Then there is a constant $L(\alpha)$, dependent only on α, such that $|Q(\zeta)| \leq L(\alpha)$.*

Proof The Lagrange interpolation formula (see section 1.9.2) gives

$$\begin{aligned} Q(z) &= \sum_{m=0}^{l} \left(Q(w_m) \prod_{0 \leq k \leq l, k \neq m} \left(\frac{w_k - z}{w_k - w_m} \right) \right) \\ &= \sum_{m=0}^{l} \left(Q(w_m) \prod_{0 \leq k \leq l, k \neq m} \left(1 - \frac{w_m - z}{w_k - z} \right)^{-1} \right), \end{aligned}$$

since the right-hand side agrees with Q at $l+1$ points. Hence, since $|Q(w_j)| \leq |Q(\zeta)|^{1/2}$,

$$|Q(\zeta)| \leq |Q(\zeta)|^{1/2} \left(\sum_{m=0}^{l} \left(\prod_{0 \leq k \leq l, k \neq m} \left| 1 - \frac{w_m - \zeta}{w_k - \zeta} \right|^{-1} \right) \right). \tag{5.6}$$

We seek a lower bound for terms of the form $|1-(w_i-\zeta)/(w_j-\zeta)|$, $i \neq j$. Set

$$\psi(u) = \frac{\phi(\eta + (u_j - \eta)u) - \phi(\eta)}{\phi(u_j) - \phi(\eta)},$$

and note that ψ is K-quasiconformal as it is a composition of ϕ with two Möbius transformations. Note also that $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(\infty) = \infty$, so ψ belongs to the family $\mathcal{F}$ of K-quasiconformal maps which fix 0, 1 and ∞. By Theorem 1.8.1, $\mathcal{F}$ is normal and compact.

We therefore need a lower bound for $|\psi(v) - 1|$, where

$$v = \frac{u_i - \eta}{u_j - \eta}.$$

It is elementary to show that $|1-v| \geq \sin\alpha = C_0 > 0$. Now, since $\mathcal{F}$ is normal and compact, every sequence in $\mathcal{F}$ has a convergent subsequence, with the limit function in $\mathcal{F}$. Suppose we can find a sequence (v_n) with $|1 - v_n| \geq C_0$ and $\phi_n \in \mathcal{F}$ such that $\phi_n(v_n) - 1 \to 0$ as $n \to \infty$. Without loss of generality, we can assume these sequences are convergent, with $v_n \to \hat{v}$ and $\phi_n \to \hat{\phi}$ (locally uniformly), with $\hat{\phi} \in \mathcal{F}$. Then $\hat{\phi}(\hat{v}) = 1$, but $|1 - \hat{v}| \geq C_0$, contradicting the fact that $\hat{\phi}(1) = 1$. Hence there is no such sequence, and therefore there is some $d(\alpha) > 0$ such that

$$\left| 1 - \frac{w_i - \zeta}{w_j - \zeta} \right| \geq d(\alpha), \tag{5.7}$$

with $d(\alpha)$ independent of ϕ, ζ and the w_j. Thus we may substitute $d(\alpha)$ into equation (5.6) to obtain the estimate

$$|Q(\zeta)| \leq \frac{(l+1)|Q(\zeta)|^{1/2}}{d(\alpha)^l},$$

and hence we must have

$$|Q(\zeta)| \leq \frac{(l+1)^2}{d(\alpha)^{2l}} = L(\alpha).$$

Lemma 5.4.2 *Let f be analytic and unbounded in D, and assume for some $k \geq 0$ the existence of a sequence (z_m) as in (5.1). Let m be large, and let the Taylor expansion of f about z_m be as in equations (5.2), (5.3) and (5.4). Let*

$$\beta_m = \min_{0 \leq j \leq k} |f^{(j)}(z_m)|^{1/2}.$$

Then we can find a path γ_m in D joining z_m to E on which $|T_m^{(j)}(z)| > \beta_m$ for all $j \in \{0, \ldots, k\}$.

Proof This is a refinement of Gordon's lemma. Obviously, the $k+1$ polynomials $T_m^{(j)}(z)$, $j \in \{0, \ldots, k\}$, each have degree no more than k. Also, recall that $T_m^{(j)}(z_m) = f^{(j)}(z_m)$ for $j \in \{0, \ldots, k\}$. Pick a positive α with $\alpha < \pi/(20k(k+1))$. Consider a particular j and take m so large that $|T_m^{(j)}(z_m)| > L(\alpha)$, where $L(\alpha)$ is as in Lemma 5.4.1. Denote $\phi^{-1}(z_m)$ by η_m, and let

$$\begin{aligned} D_{m,j} &= \{\theta \in [0, 2\pi) : \text{the ray } \arg(u - \eta_m) = \theta \text{ contains at least one point} \\ &\qquad u \text{ such that } |T_m^{(j)}(\phi(u))| \leq |f^{(j)}(z_m)|^{1/2}\}. \end{aligned}$$

Suppose n is maximal such that there exist n rays R_i, $0 \leq i \leq n-1$, emanating from η_m and separated by an angle at least α, such that on each R_i there exists a point u_i with $|T_m^{(j)}(\phi(u_i))| \leq |f^{(j)}(z_m)|^{1/2}$. By Lemma 5.4.1, it is clear that $n \leq k$.

Now suppose we can find an $(n+1)$th point u_n with $|T_m^{(j)}(\phi(u_n))| \leq |f^{(j)}(z_m)|^{1/2}$ which is not on any of these rays. By the maximality of n, u_n must be contained in one of the regions

$$\{u : |\arg(u - \eta_m) - \arg(u_i - \eta_m)| < \alpha\},$$

for $0 \leq i \leq n-1$, and so the measure of $D_{m,j}$ is at most $2k\alpha$.

Applying this argument to each of the $T_m^{(j)}$, we have that the measure of

$$D_m = \bigcup_{j=0}^{k} D_{m,j},$$

is not more than $2k(k+1)\alpha$, which is less than $\pi/10$ by the choice of α. Since the angular spread, as measured from any point in Δ, of $\overline{B(0,\frac{1}{2})}$ is at least $2\arctan\frac{1}{2} \approx 0.3\pi$, we can find a ray joining η_m to $S(0,\frac{1}{2})$ which fails to intersect with the set D_m and which forms an angle less than $\pi/20$ with the radius joining 0 to η_m. We take the ϕ-image of this ray as our path γ_m in D, thus obtaining a path satisfying condition (I)(c)(i) of Lemma 5.2.1.

Lemma 5.4.3 *There exists a constant $C > 0$ such that, for all large m, $L(\gamma_m) < C$.*

Proof Since the K-quasiconformal mapping ϕ satisfies a uniform Hölder condition on Δ with exponent $1/K$ (see [3, p.47]), we have by a theorem of Hardy and Littlewood (see, for example, [12, p.74]) that there exists a constant M_0 such that

$$|\phi'(z)| \leq M_0(1-|z|)^{-1+1/K}$$

for all $z \in \Delta$. The pre-image of γ_m is a ray R_m joining $\eta_m = \phi^{-1}(z_m)$ to $S(0,\frac{1}{2})$, forming an angle less than $\pi/20$ with the radius joining 0 to η_m. Denote the first point of intersection of R_m with $S(0,\frac{1}{2})$ by v_m. Then the acute angle μ between $[v_m,\eta_m]$ and the radius passing through v_m is, by elementary geometry, strictly less than $\pi/4$ (see Figure 5.1).

Now, for z on $[v_m,\eta_m]$, at a distance t from v_m, consider $r = |z(t)|$. By the cosine rule, we have

$$r^2 = \frac{1}{4} + t^2 + t\cos\mu, \quad \frac{dr}{dt} \geq \cos\mu > \frac{1}{2}, \tag{5.8}$$

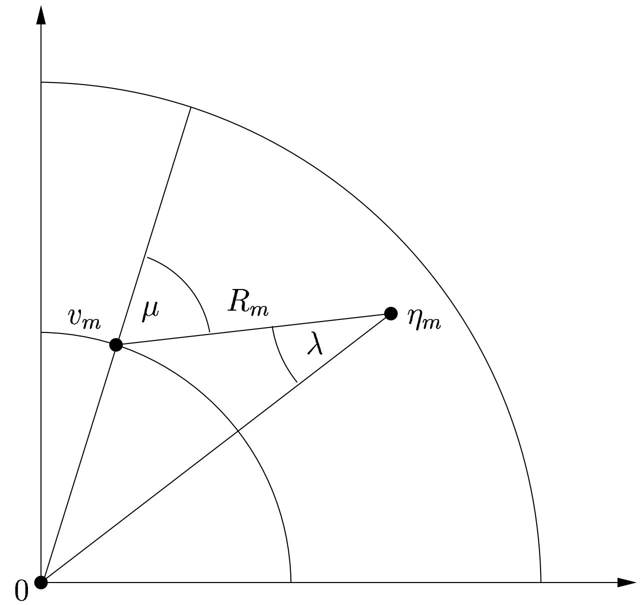

Figure 5.1: The angle λ is chosen to be no greater than $\pi/20$.

and so r increases with t. In particular, z may be parametrised in terms of r, and using (5.8) again we have

$$
\begin{aligned}
L(\gamma) &= \int_0^{|\eta_m - v_m|} |\phi'(z(t))|dt \\
&\leq 2M_0 \int_{|v_m|}^{|\eta_m|} (1-r)^{-1+1/K} dr \\
&\leq 2M_0 \int_0^1 (1-r)^{-1+1/K} dr \\
&= C,
\end{aligned}
$$

where C is a positive constant depending only on M_0 and K, so the lemma is proved.

Hence, Theorem 5.1.2 is proved, since for any k the conditions of Lemma 5.2.1(I) are satisfied if we take β_m and the paths γ_m as in Lemma 5.4.2.

Chapter 6

Normality and fixpoints of analytic functions

6.1 Introduction

A *fixpoint* of a meromorphic function f is a point z at which $f(z) = z$. It was conjectured by Gross in [19] that if f and g are non-linear entire functions, at least one of them transcendental, then the composite function $f \circ g$ has infinitely many fixpoints. The case in which f is a polynomial is proved by Rosenbloom [35] and the case in which g is a polynomial follows from an argument due to Gross and Yang [21], who observed that $f \circ g$ has infinitely many fixpoints if and only if $g \circ f$ does. The remaining case, in which f and g are both transcendental, is dealt with by Bergweiler in [5].

Gross and Osgood extend this idea to meromorphic functions in [20], showing that $f \circ g$ has infinitely many fixpoints when f is a nonlinear rational function of order greater than 2 and g is transcendental and meromorphic. Bergweiler shows in [7] that $f \circ g$ has infinitely many fixpoints when f is meromorphic and g is transcendental and entire.

Although the heuristic principle of Bloch relating properties of functions in the plane to normality criteria does not apply in general (see section 1.7), the above discussion nevertheless motivates the question of whether a family $\mathcal{G}$ of functions g analytic on a domain D and such that $f \circ g$ has no fixpoints on D is necessarily normal. If f is a polynomial of degree d at least 2, then such a family $\mathcal{G}$ is always normal, as shown by Fang and Yuan in [17], though normality can fail in the case $d = 1$, as is easily seen by taking $g_n(z) = e^{nz} + z/\alpha - \beta/\alpha$ and $f(z) = \alpha z + \beta$.

We prove a corresponding result for transcendental f.

Theorem 6.1.1 [1] *Let f be transcendental and meromorphic in the plane, and let D be a domain in $\mathbb{C}$. If $\mathbb{C}^* \setminus f(\mathbb{C}) = \emptyset$, $\{\infty\}$ or $\{\alpha, \beta\}$, where α and β are two distinct values in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, then the family*

$$\mathcal{G} = \{g : g \text{ analytic in } D, f \circ g(z) \neq z \text{ for all } z \in D\}$$

is normal in D.

Obviously, the hypotheses of Theorem 6.1.1 are always satisfied for entire f. It would be interesting to know whether a corresponding result holds for meromorphic f omitting exactly one (finite) value.

[1] A paper containing the proof of Theorem 6.1.1 has been accepted for publication and is to appear in the journal *Proceedings of the Royal Society of Edinburgh, Section A.*

6.2 A useful lemma

We make use of the following fact, and for completeness give a proof.

Lemma 6.2.1 *Let $\mathcal{F}$ be a family of functions analytic on a domain D and suppose that $\mathcal{F}$ is not normal at a point w in D but is normal on a punctured neighbourhood $E = B(w,s)\backslash\{w\}$ of w in D, for some positive s. Let (f_n) be a sequence of functions in $\mathcal{F}$ which has no locally uniformly convergent subsequence on any neighbourhood of w. Then there exists a subsequence (f_{n_k}) of (f_n) and a sequence of points (z_{n_k}) tending to w such that $f_{n_k}(z_{n_k}) = 0$ but the f_{n_k} tend to infinity locally uniformly on E.*

Proof Since $\mathcal{F}$ is normal on E, there exists a subsequence (f_{n_k}) which tends locally uniformly to a function ϕ in E, where ϕ is either analytic or identically infinite in E.

Suppose ϕ is analytic. Consider a circle $S(w,r)$, $r > 0$, contained in E; there exists an $M > 0$ such that $|\phi(z)| < M$ on $S(w,r)$. By the locally uniform convergence of the sequence (f_{n_k}), for large k we have $|f_{n_k}(z)| \leq 2M$ on this circle, and so by the maximum principle we have $|f_{n_k}(z)| \leq 2M$ in $\overline{B(w,r)}$. The local uniform boundedness of the f_{n_k} near a point implies normality there, and so to avoid a contradiction ϕ must be identically infinite in the whole of E.

Hence, for a further subsequence which we still label (f_{n_k}), we must have $0 \in f_{n_k}(B(w,r))$, for otherwise we could apply the maximum principle to $1/f_{n_k}$ to obtain a subsequence of (f_{n_k}) converging locally uniformly to infinity on $B(w,s)$. Let z_{n_k} be the nearest zero of f_{n_k} to w. Then $z_{n_k} \to w$, for if $|z_{n_k} - w| \geq 2r_1 > 0$ for infinitely many k, we can apply the maximum principle to $1/f_{n_k}$ in $B(w, r_1)$ and again obtain a subsequence tending locally uniformly to infinity on a neighbourhood of w.

6.3 Proof of Theorem 6.1.1

Let D be a domain in $\mathbb{C}$ and let f be a transcendental meromorphic function such that $f \circ g(z) \neq z$ for all $z \in D$ and all $g \in \mathcal{G}$. In view of Picard's theorem, it suffices to prove the following three lemmas.

Lemma 6.3.1 *If $\alpha \in D$ is a value taken at five or more distinct points by f, then $\mathcal{G}$ is normal at α.*

Proof Denote five distinct points at which f takes the value α by a_j, $j = 1, \ldots, 5$. Let $\varepsilon > 0$ be such that $B(\alpha, \varepsilon) \subseteq D$, and take, for each j, a small radius δ_j such that $H_j = f(B(a_j, \delta_j)) \subseteq B(\alpha, \varepsilon)$ and $\alpha \notin f(\{z : 0 < |z - a_j| < \delta_j\})$. Finally take, for each j, a $\nu_j > 0$ such that $|f(z) - \alpha| > \nu_j$ on the circle $S(a_j, \delta_j/2)$, and choose an $L > 1$ so large that $\varepsilon/L < \min \nu_j$.

If we assume that $\mathcal{G}$ is not normal at α, the Ahlfors five-island theorem (Theorem 1.7.3) implies that there exists some $g \in \mathcal{G}$ such that g maps some subdomain $U \subseteq B(\alpha, \varepsilon/L)$ conformally onto one of the five discs $B(a_j, \delta_j)$. Hence, there exists a set $V \subseteq U$ which is mapped conformally onto $B(a_j, \delta_j/2)$, with ∂V a simple closed curve mapped by g conformally onto the circle $S(a_j, \delta_j/2)$.

Now, on ∂V we have $|f \circ g(z) - \alpha| > \nu_j > \varepsilon/L$, and $f \circ g - \alpha$ has a zero in V. However, since $V \subseteq B(\alpha, \varepsilon/L)$, we have $|\alpha - z| \leq \varepsilon/L$ on ∂V, and by Rouché's theorem (Theorem 1.9.2) we have a fixpoint of $f \circ g$ in V and so in D, a contradiction.

Lemma 6.3.2 *If $\mathcal{G}$ is not normal at a point $\alpha \in D$, then f omits the value α.*

Proof By Lemma 6.3.1 and Picard's theorem, any point of non-normality of $\mathcal{G}$ must be isolated. Suppose that $\mathcal{G}$ is not normal at $\alpha \in D$ but $f(a) = \alpha$ for some point $a \in \mathbb{C}$. Let $\delta > 0$ be such that $\alpha \notin f(\{z : 0 < |z - a| < \delta\})$ and $f(B(a, \delta)) \subseteq D$. Then there exists a positive ν such that $|f(z) - \alpha| > \nu$ on $\{z : \delta/2 \leq |z - a| \leq 3\delta/4\}$.

Using Lemma 6.2.1, we can take a sequence (g_n) of functions in $\mathcal{G}$ and a sequence of points $z_n \to \alpha$ such that $g_n(z_n) = 0$ for all n, with g_n tending to infinity locally uniformly on a punctured neighbourhood $G \subseteq D$ of α. Let $0 < r < \nu$ be such that $\overline{B(\alpha, r)}$ is contained in $G \cup \{\alpha\}$. For each n, let

$$M_n = \min\{|g_n(z)| : |z - \alpha| = r\},$$

noting that $M_n \to \infty$ as $n \to \infty$. For large enough n, we have $z_n \in B(\alpha, r)$, and so by Rouché's theorem g_n takes the value a in $B(\alpha, r)$, at c, say.

Take n large, choose ρ with $\delta/2 < \rho < 3\delta/4$ such that there are no critical values of g_n on the circle $S(a, \rho)$, and let V be the component of the set

$$\{z \in B(\alpha, r) : |g_n(z) - a| < \rho\}$$

which contains c. Since n and M_n are large, we have $\partial V \subseteq B(\alpha, r)$, and by the maximum principle and the choice of ρ, ∂V is a simple closed curve.

Hence, since $|f(g_n(z)) - \alpha| > \nu > r$ on ∂V and $f \circ g_n$ takes the value α in V, by Rouché's theorem $f \circ g_n$ has a fixpoint in $B(\alpha, r)$, a contradiction.

Lemma 6.3.3 *Suppose that f omits the value $\alpha \in D$ and that f has another omitted value $\beta \in \mathbb{C}^* \backslash (f(\mathbb{C}) \cup \{\alpha\})$. Then $\mathcal{G}$ is normal at α.*

Proof Assume $\mathcal{G}$ is not normal at α. If $\beta = \infty$, the functions $(f(g(z)) - \alpha)/(z-\alpha)$, $g \in \mathcal{G}$, are analytic and omit the values 0 and 1 in $D\backslash\{\alpha\}$. Hence the family

$$\mathcal{H} = \left\{ \frac{f(g(z)) - \alpha}{z - \alpha} : g \in \mathcal{G} \right\}$$

is normal in $D\backslash\{\alpha\}$ by Montel's theorem (Theorem 1.7.2). If, on the other hand, $\beta \neq \infty$, consider the functions

$$h(z) = \frac{f(g(z)) - \alpha}{f(g(z)) - \beta}$$

for $g \in \mathcal{G}$. Clearly h is analytic and non-zero in D. If there was a point $z_0 \in D$ such that

$$h(z_0) = \frac{z_0 - \alpha}{z_0 - \beta},$$

then z_0 would be a fixpoint of $f \circ g$ in D. Hence the function

$$H(z) = \left(\frac{f(g(z)) - \alpha}{f(g(z)) - \beta} \right) \left(\frac{z - \beta}{z - \alpha} \right)$$

is analytic and omits the values 0 and 1 in $D\backslash\{\alpha, \beta\}$, and so the family

$$\mathcal{H} = \left\{ \left(\frac{f(g(z)) - \alpha}{f(g(z)) - \beta} \right) \left(\frac{z - \beta}{z - \alpha} \right) : g \in \mathcal{G} \right\}$$

is normal in $D\backslash\{\alpha, \beta\}$, again by Montel's theorem.

In each case, if we take a sequence (g_n) in $\mathcal{G}$ having no subsequence converging locally uniformly near α, we can find (using Lemma 6.2.1) a punctured neighbourhood G of α on which $\mathcal{H}$ is normal, a sequence (z_n) tending to α and a subsequence of (g_n), denoted (g_n) without loss of generality, such that $g_n \to \infty$ locally uniformly on G and $g_n(z_n) = 0$ for all n. Choose $r > 0$ such that $\overline{B(\alpha, r)} \subseteq G \cup \{\alpha\}$ and set

$$M_n = \min\{|g_n(z)| : |z - \alpha| = r\},$$

noting that $M_n \to \infty$ as $n \to \infty$. If n is large enough, we have $z_n \in B(\alpha, r)$, and hence $B(0, M_n) \subseteq g_n(B(\alpha, r))$. Denote $g_n(S(\alpha, r))$ by Γ_n, and note that the Γ_n are closed curves, arbitrarily distant from and surrounding the origin.

Let (h_n) be the sequence corresponding to (g_n) in the obvious way, depending on whether β is finite. Since $\mathcal{H}$ is normal, (h_n) contains a subsequence, denoted without loss of generality by (h_n), which tends to a limit function ψ which is either identically infinite or analytic in G.

Suppose $\psi \equiv \infty$ on G. Since $h_n \to \infty$ locally uniformly on $S(\alpha, r)$, there exists for arbitrarily large positive M an $n_0(M)$ such that, for $n \geq n_0$, $|h_n(z)| \geq M$ on $S(\alpha, r)$. If $\beta = \infty$, then we have $|f(g_n(z)) - \alpha| \geq Mr$ on $S(\alpha, r)$, whereas if β is finite we have $|1/h_n| \leq 1/M$ and so

$$\left|\frac{f(g_n(z)) - \beta}{f(g_n(z)) - \alpha}\right| \leq \frac{1}{cM}, \quad \text{so} \quad \left|\frac{\alpha - \beta}{f(g_n(z)) - \alpha}\right| \leq 1 + \frac{1}{cM}$$

for $n \geq n_0$, where c is a constant independent of n. Hence, for large n, $f(z)$ is bounded away from α on the curves Γ_n, and this contradicts Iversen's theorem (Theorem 1.6.2).

On the other hand, suppose $\psi(z)$ is analytic on G. In this case, there exists some constant L_2 such that $|\psi(z)| \leq L_2$ on $S(\alpha, r)$, and so for large n we have $|h_n(z)| \leq 2L_2$ on $S(\alpha, r)$. Hence, if $\beta = \infty$, $|f(g_n(z)) - \alpha| \leq 2L_2 r$, and if $\beta \neq \infty$,

$$\left|1 + \frac{\beta - \alpha}{f(g_n(z)) - \beta}\right| < cL_2$$

for sufficiently large n, where c is a constant independent of n. Again, f is therefore bounded away from one of its omitted values on the curves Γ_n, contradicting Iversen's theorem.

Hence $\mathcal{G}$ is normal at α.

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